(1) Let (M^n, g) be a Riemannian manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Define the *gradient* vector field ∇f on M as follows.

For any point $p \in M$ define $\nabla_p f \in T_pM$ by the formula $\langle \nabla_p f, v\rangle = df_p(v)$ for any $v \in T_pM$.

Let $x = (x_1, \dots x_n)$ be some local coordinates on M.

(a) Find the formula for ∇f in coordinates x. In other words, write $\nabla f = \sum_{i=1}^n \alpha_i(x)\frac{\partial}{\partial x_i}$ and find the formulas for $\alpha_i(x)$.

(b) Prove that $\langle \nabla x_i, \nabla x_j\rangle = g^{ij}(x)$ where $g_{ij}(x)$ is the metric tensor in coordinates x.

(2) Let (M^n, g) be a Riemannian manifold and let $\lambda > 0$ be any positive real number. Let $p \in M$ be any point and let $\sigma \subset T_pM$ be a 2-plane. Let $K^g(\sigma)$ be the sectional curvature of σ with respect to g. Consider the Riemannian metric λg.

Prove that $K^{\lambda g}(\sigma) = \frac{K^g(\sigma)}{\lambda}$

(3) Let M be the Heisenberg group, i.e the group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

identified with $\mathbb{R}^3$ via (x, y, z) coordinates.

Let g be the left invariant metric on M which coincides with the canonical metric on $\mathbb{R}^3$ at $(0, 0, 0)$. Let σ be the two-plane spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in T_0\mathbb{R}^3$. Compute the sectional curvature $K(\sigma)$ of σ.