(1) Suppose a 2×2 real matrix A has a complex eigenvalue $\lambda = \alpha + i\beta$. Let v be an eigenvector for λ. Write v as $v = u + iw$ where both u and w are real vectors.

Prove that u, w are linearly independent.

(2) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix with complex eigenvalues.

(a) Prove that $bc < 0$.

(b) In terms of a, b, c, d determine if A produces clockwise or counterclockwise spirals when plotting solutions of $y' = Ay$.
(The answer will depend on the sign of a certain expression in a, b, c, d).

(3) Consider the following linear system of ODEs

$$\begin{cases} x' = 2x + y \\ y' = -5x - 2y \end{cases}$$

Find a second degree polynomial $P(x, y)$ such that any solution of the above system satisfies

$$P(x, y) = const.$$

Hint: Use an appropriate change of coordinates.