(1) Cnsider the following linear ODE

$$ay'' + by' + cy = 0$$

Suppose its characteristic equation has a double root $\lambda$.

Show that $y = te^{\lambda t}$ is a solution of the above equation.

(2) (a) Show that the general solution of the equation

$$y'' + \omega^2 y = 0$$

can be written in the form $A\cos(\omega t - \delta)$ where $A, \delta$ are arbitrary real numbers.

*Hint:* Use the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin(\alpha)\sin(\beta)$.

(b) Let $y = -\cos t + \sqrt{3}\sin t$ be a solution of $y'' + y = 0$.

Write $y$ in the form $A\cos(t - \delta)$.

(3) Consider the following IVP:

$$\begin{cases} y'' + y = \cos(\omega t) \\ y(0) = 0 \\ y'(0) = 1/2 \end{cases}$$

a) Using the variation of parameter method find the solution of the above IVP if $\omega \neq \pm 1$.

b) Prove that for any real $t$

$$\lim_{\omega \to 1} \frac{\cos\omega t - \cos t}{\omega - 1} = -t\sin t$$

c) Let $y_\omega(t)$ be the solution of the above IVP. Prove that

$$\lim_{\omega \to 1} y_\omega(t) = y_1(t)$$

for any real $t$.

You can use the fact proved in class that $y = \frac{t\sin t}{2}$ is a particular solution of the equation $y'' + y = \cos t$.

*Hint: Use parts a) and b).*