(1) Let ϕ_t be the integral flow of the differential equation $y' = f(y)$ where $f\colon R^n \to R^n$ is C^∞.

Prove that $\phi_{t+s} = \phi_t \circ \phi_s$. You can assume that the solution of

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}$$

exists for all t for any initial condition y_0.

Hint: Use the uniqueness theorem together with fact that if $y(t)$ solves $y' = f(y)$ then so does $y(t+c)$ for any constant c. (You'll have to explain why that's true).

(2) Recall that $GL(n, R)$ is the set of all invertible real $n \times n$ matrices. It is a group with respect to matrix multiplication.

Show that for any real $n \times n$ matrix A the map $t \mapsto e^{tA}$ gives a group homomorphism from $(R, +)$ to $GL(n, R)$.

(3) Let $\phi\colon R \to GL(n, R)$ be a group homomorphism. Suppose ϕ is differentiable as a map into the space of all $n \times n$ real matrices. Show that $\phi(t) = e^{tA}$ for some $n \times n$ matrix A.

Hint: Put $A = \phi'(0)$ and verify that $y = \phi(t)$ satisfies $y' = Ay$.