(1) Suppose $x(t)$ satisfies

$$\begin{cases} x' \leq a(t)x(t) \\ x(0) = 0 \end{cases}$$

where $a(t)$ is C^1 on R and $a(t) \geq 0$. Prove that $x(t) \leq 0$ for $t \geq 0$.

Hint: Use the appropriate integrating factor $\mu(t) > 0$ to show that $\mu(t)x(t)$ is nonincreasing (think of the linear equation $x' = a(t)x$ to find $\mu(t)$.)

(2) Consider the following IVP:

$$\begin{cases} y' = y^2 + f(t) \\ y(0) = 0 \end{cases}$$

where $f(t)$ is C^1 on R and $f(t) \geq 1$ for all t.

Show that $y(t) \geq \tan t$ for all $t \geq 0$ for which $y(t)$ is defined.

Hint: Consider the system

$$\begin{cases} z' = z^2 + 1 \\ z(0) = 0 \end{cases}$$

and use (1) to show that $x = z - y$ satisfies $x(t) \leq 0$ for $t \geq 0$.

(3) Consider the following system of ODEs

$$\begin{cases} x' = -x + y^3 \\ y' = -y + x^3 \end{cases}$$

(a) Find all equilibrium points of this system and describe the behavior of the associated linearized systems.

(b) Show that any solution of the system with the initial conditions satisfying $x^2(0) + y^2(0) < 1$ exists for all $t \geq 0$.

Hint: Use the phase portrait to show that any such solution remains bounded for all t. Then apply the theorem on extending solutions of ODEs.