(1) Let $\phi\colon R \to GL(n, R)$ be a group homomorphism. Suppose ϕ is **continuous** as a map into the space of all $n \times n$ real matrices. Show that $\phi(t) = e^{tA}$ for some $n \times n$ matrix A.

Hint:

(a) Using the theorem on differentiable dependence of solutions of ODEs on parameters show that the map $A \mapsto e^A$ is a local diffeomorphism near $A_0 = 0$.

(b) Use (a) to show that there is $\epsilon > 0$ such that for any matrix A sufficiently close to I there is a well-defined $\sqrt{A}$ lying in $B(0, \epsilon)$.

(c) Show that one can similarly define $A^{p/q}$ for any integers p, q with $|p| < |q|$ and any A close to I.

(d) Use (c) to show that $\phi(t) = e^{tA}$ for some matrix A and all rational t.