Solutions to Term Test 3 Practice Test 2

(1) Give the following definitions
  (a) a k-tensor on a vector space V.
  (b) A C^r-manifold without a boundary in $\mathbb{R}^n$.

Solution

  (a) A map $T\colon \underbrace{V \times V \ldots \times V}_{k \text{ times}} \to \mathbb{R}$ is a k-tensor on V if it's linear in every variable.
  (b) A set $M \subset \mathbb{R}^n$ is a k-dimensional C^r-manifold without a boundary if for every point $p \in M$ there exists a set $U \subset M$ which is open in M, an open subset $V \subset \mathbb{R}^k$ and a C^r map $f\colon V \to \mathbb{R}^n$ such that
    (i) $f(V) = U$ and $f\colon V \to U$ is 1-1 and onto;
    (ii) $rank[df_x] = k$ for any $x \in V$;
    (iii) $f^{-1}\colon U \to V$ is continuous.

(2) Let e_1, e_2, e_3, e_4 be a basis of a 4-dimensional space V.
  Let $\omega = Alt(e_1^* \otimes e_2^* + e_3^* \otimes e_4^*)$ and $\eta = 2e_2^* + e_3^*$.
  Find $(\omega \wedge \eta)(e_2, e_3, e_4)$.

Solution

We have $\omega = \frac{1!1!}{2!}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*)$. Hence $\omega \wedge \eta = \frac{1}{2}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) \wedge (2e_2^* + e_3^*) = \frac{1}{2}(e_1^* \wedge e_2^* \wedge 2e_2^* + e_1^* \wedge e_2^* \wedge e_3^* + e_3^* \wedge e_4^* \wedge 2e_2^* + e_3^* \wedge e_4^* \wedge e_3^*) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^* + e_3^* \wedge e_4^* \wedge e_2^* = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^* + e_2^* \wedge e_3^* \wedge e_4^*$.

Therefore $\omega \wedge \eta(e_2, e_3, e_4) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^*(e_2, e_3, e_4) + e_2^* \wedge e_3^* \wedge e_4^*(e_2, e_3, e_4) = 0 + 1 = 1$.

(3) Let v be an n-dimensional vector space. Let $k \geq 1$. Find all k-tensors T on V such that both T and $|T|$ are tensors.

Solution

If both T and $|T|$ are tensors then we must have $|T|(-v_1, v_2, \ldots, v_k) = -|T|(v_1, ldots, v_k)$ for any $v_1, ldots, v_k$. But $|T| \geq 0$ so this can only happen if $T \equiv 0$.

(4) (15 pts) Let $M \subset \mathbb{R}^3$ be given by $\{x^2 + y^2 - z^2 = 0\} \cap \{x + 2y - z = 1\}$.

Show that M is a manifold and compute its dimension.

Solution

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x^2 + y^2 - z^2, x + 2y - z)$. Note that f is clearly C^∞ and $M = f^{-1}\{(0, 1)\}$.

We claim that $(0, 1)$ is a regular value of f. To see this suppose $f(x, y, z) = (0, 1)$. Then $[df_{(x,y,z)}] =$
$$\begin{pmatrix} 2x & 2y & -2z \\ 1 & 2 & -1 \end{pmatrix}$$
The only way this matrix could have rank 1 if the first row is a multiple of the second one i.e. $x = \lambda, y = 2\lambda, z = \lambda$. Then we must have $0 = x^2 + y^2 - z^2 = \lambda^2 + (2\lambda)^2 - (\lambda)^2 = 4(\lambda)^2$ and hence $\lambda = 0$ and $x = y = z = 0$. This contradicts $x + 2y - z = 1$. Thus $[df_{(x,y,z)}]$ has rank=2. Therefore $(0, 1)$ is a regular value of f and hence, M is a C^∞ manifold without boundary of dimension $3 - 2 = 1$.

(5) (15 pts) Let $U \subset \mathbb{R}^2$ be given by $\{0 < x^2 + 4y^2 < 1\}$ and $f(x, y) = \frac{1}{\sqrt{x^2+4y^2}}$.

Determine if $\int_U^{ext} f$ exists and if it does compute it.

Solution

Let $U_n = \{\frac{1}{n^2} < x^2 + 4y^2 < 1\}$ where $n > 1$ be an open exhaustion of U. Then U_n is rectifiable and f is continuous and bounded on U_n. Therefore, $\int_{U_n} f$ exists for any $n > 1$ and hence $\int_{U_n}^{ext} f$ exists and $\int_{U_n}^{ext} f = \int_{U_n} f$. Let $V_n = U_n \backslash \{(x, 0) | x > 0\}$. Then V_n

is also open and rectifiable and hence $\int_{V_n} f$ also exists. Since $f \cdot \chi_{U_n} = f \cdot \chi_{V_n}$ except on a set of measure zero this means that $\int_{V_n} f = \int_{U_n} f$.

To compute $\int_{V_n} f$ we use the change of variables $(x, y) = g(r, \theta)$ given by $x = r\cos\theta, y = \frac{1}{2} r\sin\theta$ where $1/n < r < 1, 0 < \theta < 2\pi$. Then $\det[dg] = \frac{r}{2}$ and $\int_{V_n} f = \int_0^{2\pi} \int_{1/n}^1 \frac{r}{2r} = \pi(1 - 1/n)$. We see that $\lim_{n\to\infty} \int_{U_n}^{ext} f = \lim_{n\to\infty} \pi(1-1/n) = \pi$. Since $f > 0$ on U this means that $\int_U^{ext} f$ exists and $\int_U^{ext} f = \pi$.

(6) (10 pts) Let $v_1 = (1, 1, 0), v_2 = (-1, 0, 1), v_3 = (1, 1, 1)$ and $w_1 = (0, 2, 0), w_2 = (1, 1, 0), w_3 = (-2, 1, 3)$ be two bases of $\mathbb{R}^3$. Do (v_1, v_2, v_3) and (w_1, w_2, w_3) have the same orientation?

Solution

Consider the standard basis $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. The transition matrix from (e_1, e_2, e_3) to (v_1, v_2, v_3) has columns v_1, v_2, v_3, i.e. it's given by $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. We compute $\det A = 1$. Hence, (e_1, e_2, e_3) and (v_1, v_2, v_3) have the same orientation.

Similarly, we compute that the transition matrix from (e_1, e_2, e_3) to (u_1, u_2, u_3) is $B = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and $\det B = -6$. Hence, (e_1, e_2, e_3) and (u_1, u_2, u_3) have opposite orientations. Therefore, (v_1, v_2, v_3) and (w_1, w_2, w_3) have opposite orientations.

(7) Let $M \subset \mathbb{R}^n$ be a manifold with boundary and $N \subset \mathbb{R}^m$ be a manifold without boundary. Prove that $M \times N \subset \mathbb{R}^{n+m}$ is a manifold with boundary.

Solution

Let M be k-dimensional and N be l-dimensional.

First observe the following:

Observation. If $f\colon X_1 \to Y_1, g\colon X_2 \to Y_2$ are continuous then $f \times g\colon X_1 \times X_2 \to Y_1 \times Y_2$ is also continuous. here by $f \times g$ we mean the map $f \times g(x_1, x_2) = f(x_1), g(x_2)$

Let N be k-dimensional and M be l-dimensional. Let $p \in N, q \in M$. Let $f\colon V_1 \to N$ and $g\colon V_2 \to M$ be parameterizations coming from the definition of a manifold (with boundary). Here $V_1 \subset \mathbb{R}^k$ is open and $V_2 \subset \mathbb{H}^l$ is open and $p \in f(V_1)$ and $q \in g(V_2)$. Then the map $F = f \times g\colon V_1 \times V_2 \to N \times M \subset \mathbb{R}^{n+m}$ satisfies the definition of a manifold with boundary.

Indeed, $V_1 \times V_2$ is open in $\mathbb{R}^k \times \mathbb{H}^l = \mathbb{H}^{k+l}$, the map F is obviously, smooth, 1-1 and onto $f(V_1) \times g(V_2)$. the inverse map is continuous by the Observation above.

Lastly, dF clearly has maximal rank everywhere since both df and dg do.