MAT 257Y **Solutions to Term Test 2**

(1) (15 pts) let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f_1(x,y,z) = xy + z$, $f_2(x,y,z) = e^{xz} + y^2$.

Show that the level set $\{f_1 = 1, f_2 = 2\}$ can be solved near $(1,1,0)$ as $y = y(x), z = z(x)$ and compute $\frac{\partial y}{\partial x}(1)$ and $\frac{\partial z}{\partial x}(1)$.

Solution

We compute

$$[df_{(x,y,z)}] = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x & 1 \\ ze^{xz} & 2y & xe^{xz} \end{pmatrix}$$

Hence,

$$df_{(1,1,0)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

In particular,

$$\frac{\partial f}{\partial (yz)}(1,1,0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Since this matrix is invertible, by implicit function theorem, near $(1,1,0)$ the level set $\{f_1 = 1, f_2 = 2\}$ can be written as a graph of a differentiable function $y = y(x), z = z(x)$. Also, by implicit function theorem

$$\begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = -\begin{pmatrix} \frac{\partial f_1}{\partial y}(1,1,0) & \frac{\partial f_1}{\partial z}(1,1,0) \\ \frac{\partial f_2}{\partial y}(1,1,0) & \frac{\partial f_2}{\partial z}(1,1,0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x}(1,1,0) \\ \frac{\partial f_2}{\partial x}(1,1,0) \end{pmatrix} =$$

$$= -\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(2) (18 pts) Mark True or False. **If true, give a proof. If false, give a counterexample.**

Let A be a rectangle in $\mathbb{R}^n$.

(a) Let $S \subset A$. If $bd(S)$ is rectifiable then S is rectifiable.

(b) If $S \subset A$ is rectifiable then $A \backslash S$ is also rectifiable.
(c) If $S \subset A$ has measure zero then $bd(S)$ has measure zero.

Solution

(a) **False.** For example let $S = \mathbb{Q} \cap [0,1]$. Then S is not rectifiable but $bd(S) = [0,1]$ is rectifiable.
(b) **True.** We have that $\chi_{A \backslash S} = 1 - \chi_S$ is integrable as a difference of integrable functions. Therefore, $A \backslash S$ is rectifiable.
(c) **False.** For example let $S = \mathbb{Q} \cap [0,1]$. Then S has measure zero but $bd(S) = [0,1]$ does not.

(3) (15 pts) Let $f \colon Q \to \mathbb{R}$ be continuous where Q is a rectangle in $\mathbb{R}^n$. The graph of f is the set $\Gamma_f = \{(x,y) \in \mathbb{R}^{n+1} |$ such that $x \in Q, y = f(x)\}$.
Show that Γ_f has measure zero.

Solution

Since f is continuous on Q it is bouneded on Q since Q is compact. Hence f is integrable on Q. Let $\varepsilon > 0$. Then there exists a partition P of Q such that $U(f,P) - L(f,P) < \varepsilon$. This gives $U(f,P) - L(f,P) = \sum_{A \in P} (M_A(f) - m_A(f))\text{vol}A < \varepsilon$. This in turn gives that Γ_f is covered by the rectangles $A \times [m_A(f), M_A(f)]$ where $A \in P$ with total volume $< \varepsilon$. As $\varepsilon > 0$ is arbitrary this means that Γ_f has measure zero.

(4) (10 pts) Let $\phi_1, \ldots \phi_k, \ldots ..$ be a partition of unity on an open set $U \subset \mathbb{R}^n$. Let $Q \subset U$ be a rectangle contained in U.
Prove that all but finitely many ϕ_i vanish on Q.

Solution

Let $x \in Q$. By definition of partition of unity there exists $\varepsilon_x > 0$ such that ϕ_i vanishes on $B_{\varepsilon_x}(x)$ for all $i \geq N(x)$ for some finite $N(x)$. Since Q is compact we can choose a finite cover $\{B_{\varepsilon_j}(x_j)\}_{j=1,\dots,k}$ of Q such that for every $j = 1,\dots,k$ we have that ϕ_i vanishes on $B_{\varepsilon_i}(x_i)$ for $i \geq N_j$. Let $N = \max_{j=1}^k N_j$. Then for any $i \geq N$ we have that ϕ_i vanishes on Q.

(5) (15 pts) Let $A \subset \mathbb{R}^n$ be a rectangle and let $f\colon A \to \mathbb{R}$ be integrable such that $f \geq 0$ on A and $\int_A f = 0$.

Prove that $f = 0$ almost everywhere on A.

Hint: Show that f must vanish at all its points of continuity.

Solution

Let $D \subset A$ be the set of points of continuity of f. Since f is integrable, $A \backslash D$ has measure zero. Therefore, it is sufficient to show that f vanishes on D. Let $p \in D$. Suppose $f(p) = \varepsilon > 0$. Since f is continuous at p, there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon/2$ on $B_\delta(p)$. Therefore $f(x) > \varepsilon - \varepsilon/2 = \varepsilon/2$ on $B_\delta(p)$. Choose a rectangle Q contained in $B_\delta(p)$. Then

$$\int_A f \geq \int_Q f \geq \int_Q \varepsilon/2 = \mathrm{vol}(Q) \cdot \frac{\varepsilon}{2} > 0$$

This is a contradiction and hence $f(p) = 0$.

(6) (12 pts) Let $S \subset \mathbb{R}^n$ be rectifiable and $f\colon S \to \mathbb{R}$ be continuous and bounded.

Prove that $\int_S f$ exists.

Solution

Since S is rectifiable we have that $bd(S)$ has measure zero. We know that $\mathbb{R}^n \backslash bd(S) = int(S) \cup ext(S)$. Consider the function f_S. Then $f_S(x) = 0$ for any $x \in ext(S)$ and since $ext(S)$ is open this means that f_S is continuous on $ext(S)$. Also, $f_S(x) = f(x)$ for any $x \in int(S)$ and since f is continuous on $\mathbb{R}^n$ and $int(S)$ is open this means that f_S is continuous on $int(S)$. Putting the above together we see that f_S is continuous on $\mathbb{R}^n \backslash bd(S) = int(S) \cup ext(S)$. Let

A be any rectangle containing S. It exists since S is bounded. By above, f_S is continuous on $A \cap (int(S) \cup ext(S)) = A \backslash bd(S)$. Since f (and hence f_S is bounded and $A \cap bd(S) \subset bdS$ has measure zero, f_S is integrable over A.

(7) (15 pts) Let $f(x,y) = \int_0^{x^2+y} \sqrt{e^t + x^2y^4}dt$.

Find the formulas for $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$.

You do not need to evaluate the integrals involved in the formulas.

Solution

Let $F(x,y,a) = \int_0^a \sqrt{e^t + x^2y^4}dt$. Then $f(x,y) = F(x,y,x^2+y)$. By the theorem proved in class, F is C^1 on $\mathbb{R}^3$ because $\sqrt{e^t + x^2y^4}$ is C^1. Hence f is also C^1 and by the chain rule

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} F(x,y,x^2+y) + \frac{\partial}{\partial a} F(x,y,x^2+y)2x =$$

$$= \int_0^{x^2+y} \frac{\partial}{\partial x}(\sqrt{e^t + x^2y^4})dt + (\sqrt{e^{x^2+y} + x^2y^4})2x$$

$$= \int_0^{x^2+y} \frac{2xy^4}{2\sqrt{e^t + x^2y^4}}dt + (\sqrt{e^{x^2+y} + x^2y^4})2x$$

Similarly,

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} F(x,y,x^2+y) + \frac{\partial}{\partial a} F(x,y,x^2+y) \cdot 1 =$$

$$= \int_0^{x^2+y} \frac{\partial}{\partial y}(\sqrt{e^t + x^2y^4})dt + (\sqrt{e^{x^2+y} + x^2y^4})$$

$$= \int_0^{x^2+y} \frac{4x^2y^3}{2\sqrt{e^t + x^2y^4}}dt + (\sqrt{e^{x^2+y} + x^2y^4})$$