MAT 257Y  Solutions to Term Test 2

(1) (15 pts) let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be given by \( f_1(x, y, z) = xy + z, \quad f_2(x, y, z) = e^{xz} + y^2 \).

Show that the level set \( \{ f_1 = 1, f_2 = 2 \} \) can be solved near \((1,1,0)\) as \( y = y(x), \quad z = z(x) \) and compute \( \frac{\partial y}{\partial x}(1) \) and \( \frac{\partial z}{\partial x}(1) \).

**Solution**

We compute

\[
[df(x,y,z)] = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}
\end{pmatrix} = \begin{pmatrix}
y & x & 1 \\
z e^{xz} & 2y & x e^{xz}
\end{pmatrix}
\]

Hence,

\[
df(1,1,0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}
\]

In particular,

\[
\frac{\partial f}{\partial (yz)}(1,1,0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}
\]

Since this matrix is invertible, by implicit function theorem, near \((1,1,0)\) the level set \( \{ f_1 = 1, f_2 = 2 \} \) can be written as a graph of a differentiable function \( y = y(x), z = z(x) \). Also, by implicit function theorem

\[
\begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = - \begin{pmatrix}
\frac{\partial f_1}{\partial y}(1,1,0) & \frac{\partial f_1}{\partial z}(1,1,0) \\
\frac{\partial f_2}{\partial y}(1,1,0) & \frac{\partial f_2}{\partial z}(1,1,0)
\end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x}(1,1,0) \\ \frac{\partial f_2}{\partial x}(1,1,0) \end{pmatrix}
\]

\[
= - \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

(2) (18 pts) Mark True or False. **If true, give a proof. If false, give a counterexample.**

Let \( A \) be a rectangle in \( \mathbb{R}^n \).

(a) Let \( S \subset A \). If \( bd(S) \) is rectifiable then \( S \) is rectifiable.
(b) If $S \subset A$ is rectifiable then $A \setminus S$ is also rectifiable.
(c) If $S \subset A$ has measure zero then $bd(S)$ has measure zero.

**Solution**

(a) **False.** For example let $S = \mathbb{Q} \cap [0, 1]$. Then $S$ is not rectifiable but $bd(S) = [0, 1]$ is rectifiable.
(b) **True.** We have that $\chi_{A \setminus S} = 1 - \chi_S$ is integrable as a difference of integrable functions. Therefore, $A \setminus S$ is rectifiable.
(c) **False.** For example let $S = \mathbb{Q} \cap [0, 1]$. Then $S$ has measure zero but $bd(S) = [0, 1]$ does not.

(3) (15 pts) Let $f: Q \to \mathbb{R}$ be continuous where $Q$ is a rectangle in $\mathbb{R}^n$. The graph of $f$ is the set $\Gamma_f = \{(x, y) \in \mathbb{R}^{n+1} | \text{such that } x \in Q, y = f(x)\}$.
Show that $\Gamma_f$ has measure zero.

**Solution**

Since $f$ is continuous on $Q$ it is bounded on $Q$ since $Q$ is compact. Hence $f$ is integrable on $Q$. Let $\varepsilon > 0$. Then there exists a partition $P$ of $Q$ such that $U(f, P) - L(f, P) < \varepsilon$. This gives $U(f, P) - L(f, P) = \sum_{A \in P} (M_A(f) - m_A(f))\text{vol}A < \varepsilon$. This in turn gives that $\Gamma_f$ is covered by the rectangles $A \times [m_A(f), M_A(f)]$ where $A \in P$ with total volume $< \varepsilon$. As $\varepsilon > 0$ is arbitrary this means that $\Gamma_f$ has measure zero.

(4) (10 pts) Let $\phi_1, \ldots \phi_k, \ldots$ be a partition of unity on an open set $U \subset \mathbb{R}^n$. Let $Q \subset U$ be a rectangle contained in $U$.

Prove that all but finitely many $\phi_i$ vanish on $Q$.

**Solution**

Let $x \in Q$. By definition of partition of unity there exists $\varepsilon_x > 0$ such that $\phi_i$ vanishes on $B_{\varepsilon_x}(x)$ for all $i \geq N(x)$ for some finite $N(x)$. Since $Q$ is compact
we can choose a finite cover \( \{ B_{\varepsilon_j}(x_j) \}_{j=1}^{k} \) of \( Q \) such that for every \( j = 1, \ldots, k \) we have that \( \phi_i \) vanishes on \( B_{\varepsilon_i}(x_i) \) for \( i \geq N_j \). Let \( N = \max_{j=1}^{k} N_j \). Then for any \( i \geq N \) we have that \( \phi_i \) vanishes on \( Q \).

(5) (15 pts) Let \( A \subset \mathbb{R}^n \) be a rectangle and let \( f: A \to \mathbb{R} \) be integrable such that \( f \geq 0 \) on \( A \) and \( \int_A f = 0 \).

Prove that \( f = 0 \) almost everywhere on \( A \).

*Hint:* Show that \( f \) must vanish at all its points of continuity.

**Solution**

Let \( D \subset A \) be the set of points of continuity of \( f \). Since \( f \) is integrable, \( A \setminus D \) has measure zero. Therefore, it is sufficient to show that \( f \) vanishes on \( D \). Let \( p \in D \). Suppose \( f(p) = \varepsilon > 0 \). Since \( f \) is continuous at \( p \), there exists \( \delta > 0 \) such that \( |f(x) - f(p)| < \varepsilon/2 \) on \( B_\delta(p) \). Therefore \( f(x) > \varepsilon - \varepsilon/2 = \varepsilon/2 \) on \( B_\delta(p) \). Choose a rectangle \( Q \) contained in \( B_\delta(p) \). Then

\[
\int_A f \geq \int_Q f \geq \int_Q \varepsilon/2 = \text{vol}(Q) \cdot \frac{\varepsilon}{2} > 0
\]

This is a contradiction and hence \( f(p) = 0 \).

(6) (12 pts) Let \( S \subset \mathbb{R}^n \) be rectifiable and \( f: S \to \mathbb{R} \) be continuous and bounded.

Prove that \( \int_S f \) exists.

**Solution**

Since \( S \) is rectifiable we have that \( bd(S) \) has measure zero. We know that \( \mathbb{R}^n \setminus bd(S) = \text{int}(S) \cup \text{ext}(S) \). Consider the function \( f_S \). Then \( f_S(x) = 0 \) for any \( x \in \text{ext}(S) \) and since \( \text{ext}(S) \) is open this means that \( f_S \) is continuous on \( \text{ext}(S) \). Also, \( f_S(x) = f(x) \) for any \( x \in \text{int}(S) \) and since \( f \) is continuous on \( \mathbb{R}^n \) and \( \text{int}(S) \) is open this means that \( f_S \) is continuous on \( \text{int}(S) \). Putting the above together we see that \( f_S \) is continuous on \( \mathbb{R}^n \setminus bd(S) = \text{int}(S) \cup \text{ext}(S) \). Let
A be any rectangle containing $S$. It exists since $S$ is bounded. By above, $f_S$ is continuous on $A \cap (\text{int}(S) \cup \text{ext}(S)) = A \setminus \text{bd}(S)$. Since $f$ (and hence $f_S$ is bounded and $A \cap \text{bd}(S) \subset \text{bd}S$ has measure zero, $f_S$ is integrable over $A$.

(7) (15 pts) Let $f(x,y) = \int_0^{x^2+y} \sqrt{e^t + x^2y^4} dt$.

Find the formulas for $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$.

You do not need to evaluate the integrals involved in the formulas.

Solution

Let $F(x,y,a) = \int_0^a \sqrt{e^t + x^2y^4} dt$. Then $f(x,y) = F(x,y,x^2 + y)$. By the theorem proved in class, $F$ is $C^1$ on $\mathbb{R}^3$ because $\sqrt{e^t + x^2y^4}$ is $C^1$. Hence $f$ is also $C^1$ and by the chain rule

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} F(x,y,x^2 + y) + \frac{\partial}{\partial a} F(x,y,x^2 + y)2x =$$

$$= \int_0^{x^2+y} \frac{\partial}{\partial x} (\sqrt{e^t + x^2y^4}) dt + (\sqrt{e^{x^2+y} + x^2y^4})2x$$

$$= \int_0^{x^2+y} \frac{2xy^4}{2\sqrt{e^t + x^2y^4}} dt + (\sqrt{e^{x^2+y} + x^2y^4})2x$$

Similarly,

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} F(x,y,x^2 + y) + \frac{\partial}{\partial a} F(x,y,x^2 + y) \cdot 1 =$$

$$= \int_0^{x^2+y} \frac{\partial}{\partial y} (\sqrt{e^t + x^2y^4}) dt + (\sqrt{e^{x^2+y} + x^2y^4})$$

$$= \int_0^{x^2+y} \frac{4x^2y^3}{2\sqrt{e^t + x^2y^4}} dt + (\sqrt{e^{x^2+y} + x^2y^4})$$