Solutions to Term Test 2 Practice Test 2

(1) (20 pts) Let $F(x, y)$ be given by the formula

$$F(x,y) = \int_0^y e^x \cos(t^2 x)dt$$

(a) Show that F is C^1 on $\mathbb{R}^2$.
(b) Let $c = F(0, 2)$. Compute c and prove that near $(0, 2)$ the level set $\{F(x, y) = c\}$ can be written as a graph of a differentiable function $y = g(x)$ and find $g'(0)$.

Solution

(a) Since $x \cos(t^2 x)$ is C^1 on $\mathbb{R}^2$, by a Theorem from class $\frac{\partial F}{\partial x}(x, y) = \int_0^y \frac{\partial}{\partial x}(e^x \cos(t^2 x))dt = \int_0^y (e^x \cos(t^2 x) - e^x \sin(t^2 x) \cdot t^2)dt$ is continuous on $\mathbb{R}^2$.
By the Fundamental Theorem of Calculus, we have that $\frac{\partial F}{\partial y}(x, y) = e^x \cos(y^2 x)$ is continuous on $\mathbb{R}^n$. Therefore F is C^1 on $\mathbb{R}^2$.
(b) $c = F(0, 2) = \int_0^2 e^0 \cos(t^2 \cdot 0)dt = \int_0^2 dt = 2$.
By part a) we compute $\frac{\partial F}{\partial x}(0, 2) = \int_0^2 e^0 \cos(0) - e^0 \sin(0) \cdot t^2 dt = 2$ and $\frac{\partial F}{\partial y}(0, 2) = e^0 \cos(2^2 \cdot 0) = 1 \neq 0$. By the implicit function theorem this means that near $(0, 2)$ the level set $\{F(x, y) = c\}$ can be written as a graph of a differentiable function $y = g(x)$ and $g'(0) = -\frac{\frac{\partial F}{\partial x}(0,2)}{\frac{\partial F}{\partial y}(0,2)} = -\frac{2}{1} = -2$.

(2) (10 pts) Let $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}^n$ be C^1 such that $\det([df_p]) \neq 0$ for any $p \in U$. Show that $f(U)$ is open.

Solution

Let $p \in U$. By the inverse function theorem, there exists an open set $U_p \subset U$ containing p such that

$f(U_p)$ is open and $f\colon U_p \to f(U_p)$ is 1-1 with a differentiable inverse. Therefore $f(U) = \cup_{p\in U} f(U_p)$ is open as a union of open sets.

(3) (10 pts) Let $Q \subset \mathbb{R}^n$ be a rectangle and let $f, g\colon Q \to \mathbb{R}$ be integrable. Prove that $f \cdot g$ is integrable over Q.

Solution

Since both f and g are integrable they are both bounded. this clearly implies that $f \cdot g$ is bounded. By the integrability criterion there exist sets $S_1, S_2 \subset Q$ of measure zero such that f is continuous on $Q \backslash S_1$ and g is continuous on $Q \backslash S_2$. Let $S = S_1 \cup S_2$. Then S has measure zero and both f and g are continuous on $Q \backslash S$. hence $f \cdot g$ is continuous on $Q \backslash S$. therefore $f \cdot g$ is integrable on Q.

(4) (15 pts) Mark True or False. **If true, give a proof. If false, give a counterexample.**

(a) Let (X, d) be a metric space. If $f\colon X \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq 10 \cdot d(x, y)$ for any $x, y \in X$ then f is uniformly continuous.

Answer: True. Let $\epsilon > 0$. Put $\delta = \epsilon/10$. Then if $x, y \in X$ satisfy $d(x, y) < \delta$ then $|f(x) - f(y)| \leq 10d(x, y) < 10\delta = \epsilon$ which means that f is uniformly continuous by definition of uniform continuity.

(b) Let $A \subset \mathbb{R}^n$ be a rectangle and $S \subset A$ be a subset of measure 0 then S is rectifiable.

Answer: False. For example, let $S = \mathbb{Q} \cap [0, 1]$. then S is countable and hence has measure zero but it's not rectifiable because $bd(S) = [0, 1]$ which does not have measure zero.

(c) Let $A \subset \mathbb{R}^n$ be a rectangle and $f\colon A \to \mathbb{R}$ be continuous except at finitely many points. Then f is integrable over A.

Answer: False.
For example, let $f(x)$ be given by

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Then f is continuous everywhere except at 0 but is not bounded on $[0,1]$ and hence is not integrable on $[0,1]$.

(5) (15 pts) Let A be a rectangle and $f\colon A \to \mathbb{R}$ be integrable such that f vanishes except on a set $S \subset A$ of measure 0. Prove that $\int_A f = 0$.
Hint: Show that $\overline{\int}_A f \geq 0$ and $\underline{\int}_A f \leq 0$.

Solution

Let P be any partition of A. Observe that for any rectangle $Q \in P$, Q is not entirely contained in S because Q does not have measure zero. therefore there exists $x \in Q \backslash S$. Then $f(x) = 0$. Therefore $m_Q(f) \leq 0$. Hence, $L(f,P) = \sum_{Q \in P} m_Q(f)\text{vol}Q \leq 0$. Sinc ethis is tru for any P we have that $\underline{\int}_A f \leq = \sup_P L(f,P) \leq 0$. Similarly, $\overline{\int}_A f \geq 0$. Since f is integrable we have that $\int_A f = \overline{\int}_A f = \underline{\int}_A f = 0$.

(6) (15 pts) Let $S = \{(x,y) \in \mathbb{R}^2 |$ such that $y^2 \leq x \leq 4\}$. Let $f(x,y) = y$.
Prove that f is integrable over S and compute $\int_S f$.

Solution

First observe that $bd(S)$ is the union of the interval $\{x = 4\} \times \{-2 \leq y \leq 2\}$ and the graph of $x = y^2$ for $-2 \leq y \leq 2$. Both of them have measure zero as graphs of integrable functions. Therefore, $bd(S)$ has measure zero. Since f is continuous it's integrable over S. Hence, by Fubini's theorem and we have

$$\int_S f = \int_{-2}^{2}(\int_{y^2}^{4} ydx)dy = \int_{-2}^{2}(4-y^2)ydy = \int_{-2}^{2} 4y - y^3dy =$$

$$= (2y^2 - \frac{y^4}{4})|_{-2}^{2} = 0$$

(7) (15 pts) Let S be a compact rectifiable set. Let $S \subset \cup_{i=1}^{\infty} Q_i$ for a countable collection of rectangles Q_i. Prove that $\text{vol}(S) \leq \sum_{i=1}^{\infty} \text{vol}(Q_i)$.

Hint: Reduce the problem to the case of covering by open rectangles.

Solution

Fix $\delta > 0$. for each i let Q_i' be a rectangle containing Q_i such that $Q_i \subset int(Q_i')$ and $\text{vol}Q_i' < (1+\delta)\text{vol}Q_i$. Then we have that $S \subset \cup_{i=1}^{\infty} int(Q_i')$. By compactness of S we can choose a finite subcover $S \subset \cup_{i=1}^{k} int(Q_i')$. Taking characteristic functions this means $\chi_S \leq \sum_{i=1}^{k} \chi_{Q_i'}$. Since this sum is finite and all functions in it are integrable this means that

$\text{vol}(S) = \int \chi_S \leq \int \sum_{i=1}^{k} \chi_{Q_i'} = \sum_{i=1}^{k} \int \chi_{Q_i'} = \sum_{i=1}^{k} \text{vol}(Q_i') <$
$< (1+\delta) \sum_{i=1}^{k} \text{vol}(Q_i) < (1+\delta) \sum_{i=1}^{\infty} \text{vol}(Q_i)$.

Since this is true for any $\delta > 0$ we conclude that $\text{vol}(S) \leq \sum_{i=1}^{\infty} \text{vol}(Q_i)$.