Solutions to Term Test 2 Practice Test 2

(1) (20 pts) Let \( F(x, y) \) be given by the formula
\[
F(x, y) = \int_0^y e^x \cos(t^2 x) dt
\]

(a) Show that \( F \) is \( C^1 \) on \( \mathbb{R}^2 \).

(b) Let \( c = F(0, 2) \). Compute \( c \) and prove that near \((0, 2)\) the level set \( \{F(x, y) = c\} \) can be written as a graph of a differentiable function \( y = g(x) \) and find \( g'(0) \).

Solution

(a) Since \( x \cos(t^2 x) \) is \( C^1 \) on \( \mathbb{R}^2 \), by a Theorem from class
\[
\frac{\partial F}{\partial x}(x, y) = \int_0^y \frac{\partial}{\partial x}(e^x \cos(t^2 x)) dt = \int_0^y (e^x \cos(t^2 x) - e^x \sin(t^2 x) \cdot t^2) dt
\]
is continuous on \( \mathbb{R}^2 \).

By the Fundamental Theorem of Calculus, we have that \( \frac{\partial F}{\partial y}(x, y) = e^x \cos(y^2 x) \) is continuous on \( \mathbb{R}^n \). Therefore \( F \) is \( C^1 \) on \( \mathbb{R}^2 \).

(b) \( c = F(0, 2) = \int_0^2 e^0 \cos(t^2 \cdot 0) dt = \int_0^2 dt = 2 \).

By part a) we compute \( \frac{\partial F}{\partial x}(0, 2) = \int_0^2 e^0 \cos(0) - e^0 \sin(0) \cdot t^2 dt = 2 \) and \( \frac{\partial F}{\partial y}(0, 2) = e^0 \cos(2^2 \cdot 0) = 1 \neq 0 \). By the implicit function theorem this means that near \((0, 2)\) the level set \( \{F(x, y) = c\} \) can be written as a graph of a differentiable function \( y = g(x) \) and \( g'(0) = -\frac{\frac{\partial F}{\partial x}(0, 2)}{\frac{\partial F}{\partial y}(0, 2)} = -\frac{2}{1} = -2 \).

(2) (10 pts) Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R}^n \) be \( C^1 \) such that \( \det([df_p]) \neq 0 \) for any \( p \in U \). Show that \( f(U) \) is open.

Solution

Let \( p \in U \). By the inverse function theorem, there exists an open set \( U_p \subset U \) containing \( p \) such that
\( f(U_p) \) is open and \( f: U_p \to f(U_p) \) is 1-1 with a differentiable inverse. Therefore \( f(U) = \cup_{p \in U} f(U_p) \) is open as a union of open sets.

(3) (10 pts) Let \( Q \subset \mathbb{R}^n \) be a rectangle and let \( f, g: Q \to \mathbb{R} \) be integrable. Prove that \( f \cdot g \) is integrable over \( Q \).

**Solution**

Since both \( f \) and \( g \) are integrable they are both bounded. This clearly implies that \( f \cdot g \) is bounded. By the integrability criterion there exist sets \( S_1, S_2 \subset Q \) of measure zero such that \( f \) is continuous on \( Q \setminus S_1 \) and \( g \) is continuous on \( Q \setminus S_2 \). Let \( S = S_1 \cup S_2 \). Then \( S \) has measure zero and both \( f \) and \( g \) are continuous on \( Q \setminus S \). Hence \( f \cdot g \) is continuous on \( Q \setminus S \). Therefore \( f \cdot g \) is integrable on \( Q \).

(4) (15 pts) Mark True or False. If true, give a proof. If false, give a counterexample.

(a) Let \((X, d)\) be a metric space. If \( f: X \to \mathbb{R} \) satisfies \( |f(x) - f(y)| \leq 10 \cdot d(x, y) \) for any \( x, y \in X \) then \( f \) is uniformly continuous.

**Answer:** True. Let \( \epsilon > 0 \). Put \( \delta = \epsilon / 10 \). Then if \( x, y \in X \) satisfy \( d(x, y) < \delta \) then \( |f(x) - f(y)| \leq 10d(x, y) < 10\delta = \epsilon \) which means that \( f \) is uniformly continuous by definition of uniform continuity.

(b) Let \( A \subset \mathbb{R}^n \) be a rectangle and \( S \subset A \) be a subset of measure 0 then \( S \) is rectifiable.

**Answer:** False. For example, let \( S = Q \cap [0, 1] \), then \( S \) is countable and hence has measure zero but it’s not rectifiable because \( bd(S) = [0, 1] \) which does not have measure zero.

(c) Let \( A \subset \mathbb{R}^n \) be a rectangle and \( f: A \to \mathbb{R} \) be continuous except at finitely many points. Then \( f \) is integrable over \( A \).
Answer: False.

For example, let \( f(x) \) be given by

\[
    f(x) = \begin{cases} 
        \frac{1}{x} & \text{if } x \neq 0 \\
        0 & \text{if } x = 0 
    \end{cases}
\]

Then \( f \) is continuous everywhere except at 0 but is not bounded on \([0, 1]\) and hence is not integrable on \([0, 1]\).

(5) (15 pts) Let \( A \) be a rectangle and \( f: A \to \mathbb{R} \) be integrable such that \( f \) vanishes except on a set \( S \subset A \) of measure 0. Prove that \( \int_A f = 0 \).

Hint: Show that \( \overline{\int}_A f \geq 0 \) and \( \underline{\int}_A f \leq 0 \).

Solution

Let \( P \) be any partition of \( A \). Observe that for any rectangle \( Q \in P \), \( Q \) is not entirely contained in \( S \) because \( Q \) does not have measure zero. Therefore, there exists \( x \in Q \setminus S \). Then \( f(x) = 0 \). Therefore \( m_Q(f) \leq 0 \). Hence, \( \underline{L}(f, P) = \sum_{Q \in P} m_Q(f) \text{vol} Q \leq 0 \). Since this is true for any \( P \) we have that \( \underline{\int}_A f \leq \sup_P \underline{L}(f, P) \leq 0 \). Similarly, \( \overline{\int}_A f \geq 0 \). Since \( f \) is integrable we have that \( \int_A f = \overline{\int}_A f = \underline{\int}_A f = 0 \).

(6) (15 pts) Let \( S = \{(x, y) \in \mathbb{R}^2 \mid \text{such that } y^2 \leq x \leq 4\} \). Let \( f(x, y) = y \).

Prove that \( f \) is integrable over \( S \) and compute \( \int_S f \).

Solution

First observe that \( \text{bd}(S) \) is the union of the interval \( \{x = 4\} \times \{-2 \leq y \leq 2\} \) and the graph of \( x = y^2 \) for \(-2 \leq y \leq 2\). Both of them have measure zero as graphs of integrable functions. Therefore, \( \text{bd}(S) \) has measure zero. Since \( f \) is continuous it’s integrable.
over $S$. Hence, by Fubini’s theorem and we have
\[
\int_{S} f = \int_{-2}^{2} \left( \int_{y^2}^{4} ydx \right) dy = \int_{-2}^{2} (4 - y^2) ydy = \int_{-2}^{2} 4y - y^3 dy = (2y^2 - \frac{y^4}{4})\bigg|_{-2}^{2} = 0
\]

(7) (15 pts) Let $S$ be a compact rectifiable set. Let $S \subset \bigcup_{i=1}^{\infty} Q_i$ for a countable collection of rectangles $Q_i$. Prove that $\text{vol}(S) \leq \sum_{i=1}^{\infty} \text{vol}(Q_i)$.

Hint: Reduce the problem to the case of covering by open rectangles.

Solution

Fix $\delta > 0$. for each $i$ let $Q'_i$ be a rectangle containing $Q_i$ such that $Q_i \subset \text{int}(Q'_i)$ and $\text{vol}(Q'_i) < (1 + \delta)\text{vol}(Q_i)$. Then we have that $S \subset \bigcup_{i=1}^{\infty} \text{int}(Q'_i)$. By compactness of $S$ we can choose a finite subcover $S \subset \bigcup_{i=1}^{k} \text{int}(Q'_i)$. Taking characteristic functions this means $\chi_S \leq \sum_{i=1}^{k} \chi_{Q'_i}$. Since this sum is finite and all functions in it are integrable this means that

\[
\text{vol}(S) = \int \chi_S \leq \int \sum_{i=1}^{k} \chi_{Q'_i} = \sum_{i=1}^{k} \int \chi_{Q'_i} = \sum_{i=1}^{k} \text{vol}(Q'_i) < (1 + \delta) \sum_{i=1}^{k} \text{vol}(Q_i) < (1 + \delta) \sum_{i=1}^{\infty} \text{vol}(Q_i).
\]

Since this is true for any $\delta > 0$ we conclude that $\text{vol}(S) \leq \sum_{i=1}^{\infty} \text{vol}(Q_i)$. 