MAT 257Y Solutions to Practice Term Test 2

(1) Let $f\colon R^n \to R^m$ be C^1 where $n > m$. Suppose $[df(x_0)]$ has rank m.

Show that there exists $\epsilon > 0$ such that for any $y \in B(f(x_0), \epsilon)$ there exists $x \in R^n$ such that $f(x) = y$.

Solution

By the Corollary to Implicit Function Theorem there exist an open set $U \subset R^n$, an open set $V \subset R^n$ containing x_0 and a diffeomorphism $\phi\colon U \to V$ such that $f(\phi(x, y)) = y$ for any $(x, y) \in U$. Here $x \in R^{n-m}, y \in R^m$. Let $p = (a, b \in U$ be the preimage of x_0. i.e. $\phi(p) = x_0$. Since U is open there is an $\epsilon > 0$ such that $B(p.\epsilon) \subset U$. Then for any $y \in B(f(x_0), \epsilon)$ there exists $u \in U$ such that $f(\phi(u)) = y$.

(2) Let A be a rectangle in R^n and let $S \subset A$ be a set of measure zero which is rectifiable. Show that S has content zero.

Hint: Use that $\int_A \chi_S$ exists and must be equal to zero.

Solution

Let $f = \chi_S$. Then $\int_A f$ exists and since $f = 0$ except on a set of measure 0, we must have $\int_A f = 0$ by a theorem from class. Let $\epsilon > 0$. Since $0 = \int_A f = \inf_P U(f, P)$, there exists a partition P of A such that $U(f, P) < \epsilon$. We have $U(f, P) = \sum_{Q \in P} M_f(Q) \mathrm{vol} Q$. By construction, $M_f(Q) = 0$ if $Q \cap S = \emptyset$ and $M_f(Q) = 1$ if $Q \cap S \neq \emptyset$. Therefore

$$\epsilon > U(f, P) = \sum_{Q \in P, Q \cap S \neq \emptyset} \mathrm{vol}(Q)$$

Since this sum is finite and $\epsilon > 0$ is arbitrary, this means that $content(S) = 0$.

(3) Let $f\colon [0, 1] \times [0, 1] \to R$ be continuous.

Show that

$$\int_0^1 (\int_0^x f(x,y)dy)dx = \int_0^1 (\int_y^1 f(x,y)dx)dy$$

Solution

Let $g\colon\ Q \to R$ be defined by

$$g(x,y) = \begin{cases} f(x,y) \text{ if } y \leq x \\ 0 \text{ if } y > x \end{cases}$$

Then g is clearly integrable on Q and by Fubini's theorem

$$\int_Q g = \int_0^1 (\int_0^x f(x,y)dy)dx$$

and also

$$\int_Q g = \int_0^1 (\int_y^1 f(x,y)dx)dy$$

Hence

$$\int_0^1 (\int_0^x f(x,y)dy)dx = \int_0^1 (\int_y^1 f(x,y)dx)dy$$

(4) Let $f\colon\ R^2 \to R$ be C^2.

Prove that $F(x) = \int_0^1 f(x,y)dy$ is C^2 on R.

Solution

since F is C^1, by a theorem from class we know that $F(x)$ is differentiable and $F'(x) = \int_0^1 \frac{\partial f(x,y)}{\partial x} dy$. Since $\frac{\partial f(x,y)}{\partial x}$ is continuous $F'(x)$ is also continuous by another theorem from class. Furthermore, since f is C^2 we have that $\frac{\partial f(x,y)}{\partial x}$ is C^1. Applying the same argument to $F'(x)$ we see that $F'(x)$ is C^1, i.e., F is C^2.

(5) Prove that the union of countably many sets of measure zero has measure 0.

Solution

Let $S_1, S_2, \ldots$ be a sequence of sets of measure zero. Let $S = \cup_i S_i$. Let $\epsilon > 0$. Then we can cover S_i by a countable collection of of rectangles $\mathcal{C}_i$ such that $\sum_{Q \in \mathcal{C}_i} \text{vol} Q < \frac{\epsilon}{2^i}$. Let $\mathcal{C} = \cup_i \mathcal{C}_i$. Then $\mathcal{C}$ is countable, it covers S, and $\sum_{Q \in \mathcal{C}} \text{vol} Q < \epsilon/2 + \epsilon/4 + \ldots + \epsilon/2^i + \ldots = \epsilon$. Hence S has measure zero.

(6) let $S = \{(x, y) \in R^2 | \quad x^2 + y^2 \leq 1, y \geq |x|\}$.

Compute $\int_S y$.

Solution

Note that the set $\{y = |x|\}$ intersects the set $\{x^2 + y^2 = 1\}$ when $x^2 + x^2 = 1, 2x^2 = 1, x = \pm\frac{1}{\sqrt{2}}$

We compute the integral using Fubini's Theorem

$$\int_S f = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(\int_{|x|}^{\sqrt{1-x^2}} y dy\right) dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (y^2/2|_{|x|}^{\sqrt{1-x^2}}) dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1/2 - x^2) dx = (x/2 - x^3/3)|_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{8}}\right)$$

(7) Let $A \subset R^n, B \subset R^m$ be rectangles. let $f \colon A \times B \to R$ be integrable.

Prove that there is a set $S \subset A$ of measure 0 such that for any $x \in A \backslash S$ the integral $\int_B f(x, y) dy$ exists.

Solution

Let $\mathcal{L}(x) = \underline{\int}_B f(x, y) dy$ and $\mathcal{U}(x) = \overline{\int}_B f(x, y) dy$. By Fubini's theorem $\int_{A \times B} f = \int_A \mathcal{L}(x) dx = \int_A \mathcal{U}(x) dx$. Therefore $0 = \int_A \mathcal{U}(x) - \mathcal{L}(x) dx$. Since $U(x) - \mathcal{L}(x) \geq 0$ for any x there exists a set $S \subset A$ of measure 0 such that for any $x \in A\backslash S$ we have $\mathcal{U}(x) - \mathcal{L}(x) = 0$, i.e $\int_B f(x,y)dy$ exists for any $x \in A\backslash S$.

(8) Let $f\colon [-1,1] \times [-1,1] \to R$ be a continuous function. Suppose $f(-x,y) = -f(x,y)$ for any x, y.

Prove that $\int_{[-1,1]\times[-1,1]} f = 0$.

Hint: use Fubini's theorem.

Solution

First observe that if a continuous function $g\colon [-1,1] \to R$ satisfies $g(-x) = -g(x)$ then $\int_{-1}^{1} g(x)dx = 0$.

Indeed. We change variable $x = -u$ and get

$\int_{-1}^{1} g(x)dx = \int_{1}^{-1} g(-u)d(-u) = \int_{-1}^{1} g(-u)du = \int_{-1}^{1} -g(u)du = -\int_{0}^{1} g(u)du$. Therefore,

$2\int_{-1}^{1} g(x)dx = 0$, $\int_{-1}^{1} g(x)dx = 0$

Using the above and Fubini's Theorem we get

$$\int_{[-1,1]\times[-1,1]} f = \int_{-1}^{1}(\int_{-1}^{1} f(x,y)dx)dy = \int_{-1}^{1} 0dy = 0$$