MAT 257Y **Solutions to Term Test 1**

(1) (15 pts) Let $f\colon X \to Y$ be a map between metric spaces and let $p \in X$. Suppose $\lim_{x\to p} f(x)$ exists. Prove that the limit is unique; i.e., if $\lim_{x\to p} f(x) = L_1$ and $\lim_{x\to p} f(x) = L_2$ then $L_1 = L_2$.

Solution

Suppose $L_1 \neq L_2$. Pick $\epsilon > 0$ such that $\epsilon < d(L_1, L_2)/2$. Then by the definition of limit there exists $\delta > 0$ such that for any $x \in B_\delta(p)\{p\}$ we have $d(f(x), L_1) < \epsilon$ and $d(f(x), L_2) < \epsilon$. Note that at least one such x exists because by definition of the limit p is not an isolated point. By the triangle inequality we have $d(L_1, L_2) < d(f(x), L_1) + d(f(x), L_2) < \epsilon + \epsilon = 2\epsilon < d(L_1, L_2)$. This is a contradiction and hence $L_1 = L_2$.

(2) (15 pts) Let X, Y be compact metric spaces and let $f\colon X \to Y$ be continuous, 1-1 and onto. Prove that $f^{-1}\colon Y \to X$ is continuous.

Hint: Use that a map is continuous if and only if the preimage of any closed set is closed.

Solution

Let $g = f^{-1}$. Let $C \subset X$ be closed. Then C is compact because X is compact. We have that $g^{-1}(C) = f(C)$. since an image of a compact set is compact we have that $f(C)$ is compact. Hence it is closed. We have shown that $g^{-1}(C)$ is closed for any closed $C \subset X$. Therefore g is continuous.

(3) (15 pts) Let X be a metric space. Let $A \subset X$ satisfies $A \cap bd(A) = \emptyset$. Prove that A is open.

Solution

Let $a \in A$. Note that for any $\epsilon > 0$ we have that $B_\epsilon(a) \cap A$ contains a and is therefore non-empty. Since $a \notin bd(A)$ this means that there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(a) \cap X \backslash A = \emptyset$ (otherwise $a \in bd(A)$).

Therefore $B_{\epsilon_0} \subset A$. Since a was arbitrary this means that A is open.

(4) (15 pts) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Are the following statements true or false? Prove if true and give a counterexample if false.
(a) If $A \subset \mathbb{R}^n$ is closed then $f(A)$ is closed.
(b) If $A \subset \mathbb{R}^n$ is bounded then $f(A)$ is bounded.

Solution

(a) **False**. For example take $f(x) = e^x$ and $A = (-\infty, 0]$. Then A is closed but $f(A) = (0, 1]$ is not.
(b) **True.** Since A is bounded it's contained in a closed ball $\bar{B}_R(0)$ for some finite R. Since $\bar{B}_R(0)$ is closed and bounded, it is compact and hence its image is compact and therefore bounded. Since $f(A) \subset f(\bar{B}_R(0))$ it is also bounded.

(5) (15 pts) Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Find the expressions for partial derivatives of the following functions
(a)

$$f(x, y) = \int_{x^2+y}^{xy^3} g(t)dt$$

(b) $h(x, y) = \sin(g(x^2 + y^3))$

Solution

(a) Let $G(u, v) = \int_u^v g(t)dt$ and $H(x, y) = (x^2 + y, xy^3)$. Then $f = G \circ H$ and by the chain rule we get

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial G}{\partial u}(H(x, y))\frac{\partial u}{\partial x}(x, y) + \frac{\partial G}{\partial v}(H(x, y))\frac{\partial v}{\partial x}(x, y)$$

$$= g(xy^3)y^3 - g(x^2 + y) \cdot 2x$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial G}{\partial u}(H(x,y))\frac{\partial u}{\partial y}(x,y) + \frac{\partial G}{\partial v}(H(x,y))\frac{\partial v}{\partial y}(x,y)$$

$$= g(xy^3)3xy^2 - g(x^2+y)$$

(b) By the chain rule we get

$$\frac{\partial h}{\partial x}(x,y) = \cos(g(x^2+y^3)\cdot g'(x^2+y^3)\cdot 2x$$

and

$$\frac{\partial h}{\partial y}(x,y) = \cos(g(x^2+y^3)\cdot g'(x^2+y^3)\cdot 3y^2$$

(6) (15 pts) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at p. Prove that f is continuous at p.

Solution

Let $A = df_p$ and let

$$F(h) = \begin{cases} \frac{f(p+h)-f(p)-A(h)}{|h|} \text{ if } h \neq 0 \\ 0 \text{ if } h = 0 \end{cases}$$

Then F is continuous at 0 by definition of differentiability and $f(p+h) - f(p) = F(h)\cdot|h| + A(h)$ is continuos in h at 0 as a product and sum of continuous functions. Continuity of $f(p+h)-f(p)$ at $h=0$ is clearly equaivalent to continuity of f at p.

(7) (10 pts) Let $U \subset \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}^n$ be C^1. Suppose $\det[df_x] \neq 0$ for any $x \in U$.
Prove that $f(U)$ is open.

Solution

Let $q \in f(U)$. Then $q = f(p)$ for some $p \in U$. By inverse function theorem there exists an open $U' \subset U$

containing p such that $f(U')$ is open. Hence it contains $B(q, \epsilon)$ for some $\epsilon > 0$. Since $B_\epsilon(q) \subset f(U') \subset f(U)$ this means that $f(U)$ is open.