MAT 257Y Solutions to Practice Test 2

(1) Let (X, d) be a metric space. Let $A \subset X$ be a compact subset. Using only the definition of compactness prove that A is closed.

Solution

Let $U = X \backslash A$. We need to show that U is open. Let $p \in U$. Let $U_n = \{x \in X|$ such that $d(x, p) > 1/n\}$. Then U_n is open. Indeed, let $x \in U_n$. Then $d(x, p) > 1/n$. Choose $\epsilon > 0$ such that $d(x, p) - \epsilon > 1/n$. Then $B(x, \epsilon) \subset U_n$ by the triangle inequality. Hence U_n is open. Next note that $\cup_{n \in \mathbb{N}} U_n = X \backslash \{p\}$ covers A. By compactness of A we can find a finite subcover $U_{n_1}, \ldots, U_{n_k}$ covering A where $n_1 < n_2 < \ldots < n_k$. Then $\cup_{j=1}^{k} U_{n_j} = U_{n_k}$ contains A. This means that $B(p, \frac{1}{n_k}) \subset U$ and hence U is open.

(2) Let $f \colon X \to \mathbb{R}$ be continuous at $a \in X$. Prove that there exists $\delta > 0$ such that f is bounded on $B(a, \delta)$.

Solution

Choose any $\epsilon > 0$. For example, take $\epsilon = 1$. By definition of continuity, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for any $x \in B(a, \delta)$. This means that for any $x \in B(a, \delta)$ we have $|f(x)| < |f(a)| + \epsilon$. $\square$

(3) Mark True or False. **If True give a proof, if False give a counterexample.**

Let (X, d) be a metric space. Let $A, B \subset X$ be subsets in X.

(a) $ext(A)$ is open;

(b) $int(A \cup B) = int(A) \cup int(B)$.

Solution

(a) **True.** Let $p \in ext(A)$. By definition of $ext(A)$ this means that there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset X \backslash A$. We claim that $B(p, \epsilon) \subset ext(A)$.

Let $x \in B(p, \epsilon)$. Tthen $d(p, x) < \epsilon$. Choose $\delta > 0$ such that $d(p, x) + \delta < \epsilon$. Then, by the triangle inequality, $B(x, \delta) \subset B(p, \epsilon) \subset X \backslash A$. This means that $B(p, \epsilon) \subset ext(A)$ and hence $ext(A)$ is open.

(b) **False.** For example, take $X = \mathbb{R}, A = [0, 1]$ and $B = [1, 2]$. then $int(A) = (0, 1), int(B) = (1, 2)$ and $int(A \cup B) = (0, 2) \neq (0, 1) \cup (1, 2)$.

(4) Find expressions for the partial derivatives of the following functions

(a) $F(x, y) = \int_{k^2(x)h(y)}^{1} g(t)dt$

(b)
$$f(x, y) = \int_{x}^{\int_x^y g(t)dt} g(t)dt$$
Hint: put $F(x, y) = \int_x^y g(t)dt$ and express f as a composition.

(c) $f(x, y) = \ln([\sin(x + y^2)]^{\cos 2x})$

Solution

(a) Let $a(x, y) = k^2(x)h(y)$ and $f(a) = \int_a^1 g(t)dt$. Then $F(x, y) = f(a(x, y))$. Note that $f'(a) = -g(a)$ by the Fundamental Theorem of Calculus. By the chain rule,

$$\frac{\partial F}{\partial x}(x, y) = \frac{\partial f}{\partial a}(a(x, y)) \cdot \frac{\partial a}{\partial x} = f'(k^2(x)h(y)) \cdot 2k(x)h(y) =$$
$$= -g(k^2(x)h(y)) \cdot 2k(x)k'(x)h(y)$$

Similarly,

$$\frac{\partial F}{\partial y}(x, y) = -g(k^2(x)h(y)) \cdot k^2(x)h'(y)$$

(b) put $F(x, y) = \int_x^y g(t)dt$. By the fundamental theorem of calculus we have

$$\frac{\partial F}{\partial x}(x, y)(x, y) = -g(x) \text{ and } \frac{\partial F}{\partial y}(x, y)(x, y) = g(y)$$

We also have that $f(x,y) = F(x, F(x,y))$. Therefore, by the chain rule we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial F}{\partial x}(x, F(x,y))\frac{\partial x}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x, F(x,y))\frac{\partial F}{\partial x}(x,y) =$$

$$= -g(x)\cdot 1 + g(F(x,y))\cdot(-g(x)) = -g(x) - g(\int_x^y g(t)dt)g(x)$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial F}{\partial x}(x, F(x,y))\frac{\partial x}{\partial y}(x,y) + \frac{\partial F}{\partial y}(x, F(x,y))\frac{\partial F}{\partial y}(x,y) =$$

$$= -g(x)\cdot 0 + g(F(x,y))g(y) = g(\int_x^y g(t)dt)g(y)$$

(c) First we simplify $f(x,y) = \ln([\sin(x+y^2)]^{\cos 2x}) = (\cos 2x)\ln(\sin(x+y^2))$

Then we compute

$\frac{\partial f}{\partial x}(x,y) = -2(\sin 2x)\ln(\sin(x+y^2)) + (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2)$

$\frac{\partial f}{\partial y}(x,y) = (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2)(2y)$

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

(a) Is f continuous at $(0,0)$?
(b) Do $D_1 f$ and $D_2 f$ exist at $(0,0)$?
(c) Is f differentiable at $(0,0)$?

Solution

(a) Is f continuous at $(0,0)$? We can rewrite $f(x,y) = x \cdot \frac{y}{\sqrt{x^2+y^2}}$ where $\lim_{(x,y)\to 0} x = 0$ and $|\frac{y}{\sqrt{x^2+y^2}}| \leq 1$. Therefore,

$$\lim_{(x,y)\to 0} f(x,y) = 0 = f(0,0)$$

Which means that f is continuous at $(0,0)$.

Answer: Yes.

(b) Do $D_1 f$ and $D_2 f$ exist at $(0,0)$?

Observe that $f(x,0) = 0$ and $f(0,y) = 0$ for any x and any y. Therefore $D_1 f(0,0) = 0 = D_2 f(0,0)$.

Answer: Yes, bothe partial derivatives exist at $(0,0)$.

(c) Is f differentiable at $(0,0)$? If f were differentiable at 0 then all directional derivatives at $(0,0)$ would exist too. Let $v = (1,1)$ Then $D_v f(0,0) = f(tv)'(0)$ but $f(tv) = \frac{t^2}{\sqrt{t^2+t^2}} = \frac{|t|^2}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}}|t|$ has no derivative at 0. Hence $D_v f(0,0)$ does not exist and hence f is not differentiable at $(0,0)$.

Answer: No.

(6) (10 pts) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (xy, e^x + y)$.

Show that there exists an open set U containing $(0,1)$ such that $V = f(U)$ is open, f is 1-1 on U and $g = f^{-1}\colon V \to U$ is differentiable on V.

Compute $dg_{(0,2)}$.

Solution

F is clearly C^1 on $\mathbb{R}^2$. We compute the matrix of $df_{(x,y)}$. It is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ e^x & 1 \end{pmatrix}$$

Therefore, the matrix of $df_{0,1)}$ is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

It has det $= 1 \neq 0$ and hence it's invertible. By the Inverse Function Theorem, there exists an open set U containing $(0,1)$ such that $V = f(U)$ is open, f is

1-1 on U and $g = f^{-1}\colon V \to U$ is differentiable on V. Observe that $f(0,1) = (0,2)$. Again, by the Inverse Function theorem, the matrix of $dg_{(0,2)}$ is given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

(7) Let $M(n)$ be the set of all real $n \times n$ matrices identified with $\mathbb{R}^{n^2}$. Let $O(n) \subset M(n)$ be the set of all orthogonal matrices. Recall that an $n \times n$ matrix is called orthogonal if $A \cdot A^t = A^t \cdot A =$Id where A^t is the transpose of A.
(a) Prove that $O(n)$ is closed.
(b) Prove that $O(n)$ is bounded.

Solution

(a) Consider the map $f\colon M(n) \to M(n)$ given by $f(A) = A \cdot A^t$. Then f is clearly continuous as the entries of $f(A)$ are polynomials in A_{ij}'s. By definition, $O(n) = f^{-1}(\{Id\})$. Since $\{Id\} \subset M(n)$ is closed we conclude that $O(n)$ is also closed as a preimage of a closed set under a continuous map.

(b) Note that $||A|| = \sqrt{\sum_{i,j} A_{i,j}^2}$ for any $A \in M(n)$. Let $A \in O(n)$. We are given that $A \cdot A^t = Id$. Observe that the i-th column of A^t is the ith row of A. Multiplying the i-th row of A by the i-th column of A^t this gives $1 = \sum_j A_{i,j}^2$. Adding the above equations for all i's this gives $\sum_i \sum_j A_{i,j}^2 = n$. This means that $||A|| = \sqrt{n}$ for any $A \in O(n)$ and hence $O(n)$ is bounded.

Note that taken together (a) and (b) mean that $O(n)$ is compact.