Practice Term Test 2

(1) Let (X,d) be a metric space. Let $A \subset X$ be a compact subset. Using only the definition of compactness prove that A is closed.

(2) Let $f\colon X \to \mathbb{R}$ be continuous at $a \in X$. Prove that there exists $\delta > 0$ such that f is bounded on $B(a,\delta)$.

(3) Mark True or False. **If True give a proof, if False give a counterexample.**

Let (X,d) be a metric space. Let $A, B \subset X$ be subsets in X.

(a) $ext(A)$ is open;

(b) $int(A \cup B) = int(A) \cup int(B)$.

(4) Find expressions for the partial derivatives of the following functions

(a) $F(x,y) = \int_{k^2(x)h(y)}^{1} g(t)dt$

(b)

$$f(x,y) = \int_{x}^{\int_x^y g(t)dt} g(t)dt$$

Hint: put $F(x,y) = \int_x^y g(t)dt$ and express f as a composition.

(c) $f(x,y) = \ln([\sin(x+y^2)]^{\cos 2x})$

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

(a) Is f continuous at $(0,0)$?

(b) Do $D_1 f$ and $D_2 f$ exist at $(0,0)$?

(c) Is f differentiable at $(0,0)$?

(6) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (xy, e^x + y)$.

Show that there exists an open set U containing $(0,1)$ such that $V = f(U)$ is open, f is 1-1 on U and $g = f^{-1}\colon V \to U$ is differentiable on V.

Compute $dg_{(0,2)}$.

(7) Let $M(n)$ be the set of all real $n \times n$ matrices identified with $\mathbb{R}^{n^2}$. Let $O(n) \subset M(n)$ be the set of all orthogonal matrices. Recall that an $n \times n$ matrix is called orthogonal if $A \cdot A^t = A^t \cdot A =$Id where A^t is the transpose of A.

(a) Prove that $O(n)$ is closed.

(b) Prove that $O(n)$ is bounded.