MAT 257Y Solutions to Practice Term Test 1

(1) Find the partial derivatives of the following functions

(a) $F(x,y) = f(g(x)k(y), h(x) + 2k(y))$

(b) $f(x,y,z) = \sin(x \sin(y \sin z))$

(c) $f(x,y,z) = x^{yz^2}$

Solution

(a) Let $f = f(t_1, t_2)$. Then, by the chain rule we have

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial t_1}(g(x)k(y), h(x) + 2k(y)) \cdot \frac{\partial (g(x)k(y))}{\partial x} +$$

$$+ \frac{\partial f}{\partial t_2}(g(x)k(y), h(x) + 2k(y)) \cdot \frac{\partial (h(x) + k(y))}{\partial x} =$$

$$= \frac{\partial f}{\partial t_1}(g(x)k(y), h(x)+2k(y)) \cdot g'(x)k(y) + \frac{\partial f}{\partial t_2}(g(x)k(y), h(x)+2k(y)) \cdot h'(x)$$

Similarly,

$$\frac{\partial F}{\partial x}(x,y) =$$

$$= \frac{\partial f}{\partial t_1}(g(x)k(y), h(x)+2k(y)) \cdot g(x)k'(y) + \frac{\partial f}{\partial t_2}(g(x)k(y), h(x)+2k(y)) \cdot 2k'(y)$$

(b) $\frac{\partial f}{\partial x}(x,y,z) = (\cos(x \sin(y \sin z)))(\sin(y \sin z))$

$\frac{\partial f}{\partial y}(x,y,z) = (\cos(x \sin(y \sin z)))(x \cos(y \sin z)) \sin z$

$\frac{\partial f}{\partial z}(x,y,z) = (\cos(x \sin(y \sin z)))(x \cos(y \sin z)) y \cos z$

(c) First, we rewrite $f(x,y,z)$ as $f(x,y,z) = (e^{\ln x})^{yz^2} = e^{(\ln x)yz^2}$

$\frac{\partial f}{\partial x}(x,y,z) = (e^{(\ln x)yz^2})\frac{yz^2}{x} = (x^{yz^2})\frac{yz^2}{x}$

$\frac{\partial f}{\partial y}(x,y,z) = (e^{(\ln x)yz^2})(\ln x)z^2 = (x^{yz^2})(\ln x)z^2$

$\frac{\partial f}{\partial z}(x,y,z) = (e^{(\ln x)yz^2})(\ln x)y(2z) = (x^{yz^2})(\ln x)y(2z)$

(2) give an example of a nonempty set $A \subset \mathbb{R}$ such that the set of limit points of A is the same as the set of boundary points of A.

Solution

Let $A = Q$. Then $\mathbb{R} = LimA = bd(A)$.

(3) Let $A, B \subset \mathbb{R}^n$ be compact.

Prove that the set $A + B = \{a + b | a \in A, b \in B\}$ is compact.

Solution

Consider the map $f \colon \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x, y) = x + y$. This map is linear and hence continuous. By construction, $A + B = f(A \times B)$. $A \times B$ is compact as a product of two compact sets and hence $A + B = f(A \times B)$ is also compact as an image of a compact set under a continuous map.

(4) Show that the intersection of an arbitrary collection of closed sets is closed.

Solution

Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of closed sets in a metric space X. Let $U_\alpha = X\backslash A_\alpha$. Then U_α is open.

We have

$$X\backslash \cap_\alpha A_\alpha = \cup_\alpha U_\alpha$$

is open as a union of open sets. Hence $\cap_\alpha A_\alpha$ is closed.

(5) Show that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f^{-1}(A)$ is closed for any closed $A \subset \mathbb{R}^m$.

Solution

Let f be continuous.

Suppose $A \subset \mathbb{R}^m$ is closed. Then $\mathbb{R}^m\backslash A$ is open. By continuity of f this implies that $f^{-1}(\mathbb{R}^m\backslash A)$ is open. It's easy to see that $f^{-1}(\mathbb{R}^m\backslash A) = \mathbb{R}^n\backslash f^{-1}(A)$. hence $f^{-1}(A)$ is closed. The reverse implication is proved similarly.

(6) Let $GL(n,\mathbb{R})$ be the set of all $n\times n$ invertible matrices identified with $\mathbb{R}^{n^2}$.

Show that $GL(n,\mathbb{R})$ is open in $\mathbb{R}^{n^2}$.

Solution

Let $f\colon \mathbb{R}^{n^2} \to \mathbb{R}$ be given by $f(A) = \det A$ where A is an $n \times n$ matrix. Then f is continuous since it's a polynomial function. Since a matrix A is invertible iff $\det A \neq 0$ we have that $GL(n,\mathbb{R}) = f^{-1}((-\infty,0)\cup (0,\infty))$. Hence it is open as a preimage of an open set under a continuous map.

(7) Let $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula $f_1(x,y) = x + y + y^3 + 1,\ f_2(x,y) = xe^y + 2$

Show that there exists an open set U containing $(0,0)$ such that $f\colon U \to f(U)$ is a bijection and f^{-1} is differentiable on $f(U)$ and compute $df^{-1}(1,2)$.

Solution

Clearly f is differentiable everywhere. we compute $\frac{\partial f_1}{\partial x}(x,y) = 1$, $\frac{\partial f_1}{\partial y}(x,y) = 1+3y^2$, $\frac{\partial f_2}{\partial x}(x,y) = e^y$, $\frac{\partial f_2}{\partial y}(x,y) = xe^y$

Therefore

$$[df(0,0)] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix has det $= -1 \neq 0$. $f(0,0) = (1,2)$. hence, by the inverse function theorem, there exists an open set U containing $(0,0)$ such that $f\colon U \to f(U)$ is a bijection and f^{-1} is differentiable on $f(U)$ and $[df^{-1}(1,2)] = [df(0,0)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

(8) Let $f(x,y) = x^y$ be defined on $U = \{(x,y) | x > 0\}$.

Verify that

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

Solution

First we rewrite $f(x,y) = e^{(\ln x)y}$. we compute $\frac{\partial f}{\partial x}(x,y) = e^{(\ln x)y}\frac{y}{x}$, $\frac{\partial f}{\partial y}(x,y) = e^{(\ln x)y}\ln x$. Hence $\frac{\partial^2 f}{\partial x \partial y}(x,y) = e^{(\ln x)y}\frac{y}{x}\ln x + e^{(\ln x)y}\frac{1}{x}$ and $\frac{\partial^2 f}{\partial y \partial x}(x,y) = e^{(\ln x)y}\ln x\frac{y}{x} + e^{(\ln x)y}\frac{1}{x}$. Thus

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$