(1) Let X be a metric space and let $f\colon X \to \mathbb{R}$ be a bounded function. Let $p \in X$ and $\delta > 0$. Define $\nu(f, p.\delta) = \sup_{x \in B_\delta(p)} f(x) - \inf_{x \in B_\delta(p)} f(x)$. Further define $\nu(f, p) = \inf_{\delta > 0} \nu(f, p, \delta)$. $\nu(f, p)$ is called the *oscillation* of f at p.

Prove that f is continuous at p if and only if $\nu(f, p) = 0$.

(2) Let A be a rectangle in $\mathbb{R}^n$ and let $f\colon A \to \mathbb{R}$ be a bounded function. Let P be a partition of A. Prove that f is integrable on A iff f is integrable over each $Q \in P$ in which case

$$\int_A f = \sum_{Q \in P} \int_Q f$$

(3) Let A be a rectangle in $\mathbb{R}^n$ and let $f\colon A \to \mathbb{R}$ be a bounded function. Suppose f is integrable on A, $f \geq 0$ on A and $\int_A f = 0$. Prove that $f = 0$ almost everywhere on A.

(4) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that f is differentiable on $[0, 1]$ but f' is not integrable on $[0, 1]$.

Hint: Construct a function differentiable on $[0, 1]$ such that it's derivative is not bounded on $[0, 1]$.

(5) Let $f\colon Q \to \mathbb{R}$ be integrable.

Prove that $|f|$ is integrable and $|\int_Q f| \leq \int_Q |f|$.

(6) Let $f\colon [0, 1] \to [01]$ be integrable. Consider the following function $f\colon Q = [0, 1] \times [0, 1] \to \mathbb{R}$

$$F(x, y) = \begin{cases} 1 \text{ if } y < f(x) \\ 0 \text{ if } y \geq f(x) \end{cases}$$

Prove that F is integrable over Q and $\int_Q F = \int_0^1 f$.

(7) Let $f\colon [0, 1] \times [0, 4] \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} xy^2 \text{ if } y < x^2 \\ x + 2y \text{ if } y \geq x^2 \end{cases}$$

Verify that f is integrable and compute $\int_Q f$ in two different ways using Fubini's theorem.

(8) Let Q be a rectangle in $\mathbb{R}^n$. Let $S \subset Q$. consider the characteristic function of S on Q given by

$$\chi_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}$$

prove that $\chi_S(x)$ is integrable if and only if $bd(S)$ has measure 0.

Extra Credit: Give an example of two functions $f, g\colon [0, 1] \to [0, 1]$ such that f is continuous, g is integrable but $g \circ f$ is not integrable.