(1) Let A be a rectangle in $\mathbb{R}^n$ and let $Q \subset A$ be a subrectangle. Let $f\colon A \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{cases}$$

Using the definition of integrability prove that $\int_A f$ exists and $\int_A f = \text{vol(Q)}$.

(2) A set $S \subset \mathbb{R}^n$ is said to have *content* zero if for any $\epsilon > 0$ there exists a *finite* collection of rectangles Q_i covering S such that $\sum_i \text{vol(Q}_\text{i}) < \epsilon$.
   (a) Show that if $S \subset Q \subset \mathbb{R}^n$ has content zero then any bounded function $f\colon Q \to \mathbb{R}$ such that $f(x) = 0$ if $x \notin S$ is integrable over Q and $\int_Q f = 0$.
   (b) Let $S \subset Q \subset \mathbb{R}^n$ have content zero. let $f, g\colon Q \to \mathbb{R}$ be bounded and satisfy $f(x) = g(x)$ if $x \notin S$. Prove that $\int_Q f$ exists if and only if $\int_Q g$ exists and if they both exist $\int_Q f = \int_Q g$.
   (c) Show that a finite union of sets of content zero has content zero.
   (d) Let Q be a rectangle in $\mathbb{R}^n$ show that $bd(Q)$ has content zero.
   (e) Show that if S has content zero then its closure $Cl(S)$ also has content zero.
   (f) Show that $S = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$ does not have content zero. Here $\mathbb{Q}$ is the set of rational numbers.

(3) Let $f\colon Q \to \mathbb{R}$ be integrable over Q. Let $c \in \mathbb{R}$ be a constant. Prove that cf is also integrable over Q and $\int_Q cf = c \int_Q f$.

(4) Let $f_1, f_2\colon Q \to \mathbb{R}$ be integrable over Q. Prove that $f_1 + f_2$ is also integrable over Q and $\int_Q (f_1 + f_2) = \int_Q f_1 + \int_Q f_2$.

(5) Let $f\colon [0, 1] \to R$ be defined as follows

$$f(x) = \begin{cases} 1/q \text{ if } x = p/q \text{ where p,q are positive integers with no common factor} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_{[0,1]} f$.

(6) Let $f\colon Q \to \mathbb{R}$ be integrable where Q is a rectangle in $\mathbb{R}^n$. The graph of f is the set $\Gamma_f = \{(x, y) \in R^{n+1} |$ such that $x \in Q, y = f(x)\}$.
Show that Γ_f has measure zero.
Hint: Use the definition of integrability of f.