(1) Let $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the function $f(x,y) = \frac{x^p}{p} + \frac{y^q}{q} - xy$ defined on the set $A = \{x \geq 0, y \geq 0\}$. Assuming that $f$ has a minimum on $A$ find it.

(2) Let $M(2,2)$ be the space of $2 \times 2$ matrices identified with $\mathbb{R}^4$. Let $f: M(2,2) \to M(2,2)$ be given by $f(A) = A^2$. Let $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $f(A_0) = Id$. Is it true that there is an open set $U$ containing $A_0$ such that $f$ is 1-1 on $U$, $f(U)$ is open and $f^{-1}$ is differentiable on $f(U)$?

(3) Finish the proof of the following statement from class. Let $f: GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ be given by $f(A) = A^{-1}$. Let $A_0$ be the identity matrix. Prove that $df_{A_0}(X) = -X$ for any $n \times n$ matrix $X$.

(4) Give a careful proof of the following statement from class. Let $f(r,\theta) = (r \cos \theta, r \sin \theta)$ be defined on $U = \{1 < r < 2, 0 < \theta < 2\pi\}$. Prove that

(a) $f$ is 1–1 on $U$;
(b) $f(U) = V = \{1 < x^2 + y^2 < 4\}\setminus\{(x,0) \text{ with } 1 < x < 2\}$;
(c) $g = f^{-1}$: $V \to U$ is $C^1$;
(d) $||dg|\|_L$ is bounded on $V$ but $g$ is not $L$-Lipschitz on $V$ for any $L$.

Extra Credit Problem (to be written up and submitted separately)

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function such which is continuous in each variable, i.e. for every $x \in \mathbb{R}$ the function $y \mapsto f(x,y)$ is continuous and for every $y \in \mathbb{R}$ the function $x \mapsto f(x,y)$ is continuous.

(1) Give an example of such an $f$ which is not everywhere continuous on $\mathbb{R}^2$;

(2) Suppose we also assume in addition that for any compact set $K \subset \mathbb{R}^2$ its image $f(K)$ is compact. Prove that $f$ is continuous on $\mathbb{R}^2$. 