(1) Let $V \subset \mathbb{R}^n$ be a vector subspace of dimension $n-1$. Let $N \in \mathbb{R}^n$ be a nonzero vector normal to V. Let $v_1, \ldots, v_{n-1}$ be a basis of V. We'll say that $(v_1, \ldots, v_{n-1})$ is a positive basis with respect to the orientation of V induced by N if $\det(N, v_1, \ldots, v_{n-1}) > 0$.

Prove that this defines a well-defined orientation of V. in other words, suppose $(u_1, \ldots, u_{n-1})$ be another basis of V.

Prove that $(u_1, \ldots, u_{n-1})$ and $(v_1, \ldots, v_{n-1})$ have the same orientation if and only if $\det(N, v_1, \ldots, v_{n-1})$ and $\det(N, u_1, \ldots, u_{n-1})$ have the same sign.

(2) Let $M = \{x^2 + y^2 + z^2 \leq 1\}$ in $\mathbb{R}^3$ with the orientation coming from the canonical orientation on $\mathbb{R}^3$. Consider the induced orientation on ∂M and find a positive basis of $T_p \partial M$ at $p = (1, 0, 0)$.

Further, let $N = S^2_+ = \{(x, y, z)|$ such that $x^2 + y^2 + z^2 = 1$ and $z \geq 0\}$. Consider the orientation on N coinciding with the orientation on $S^2 = \partial M$. Consider ∂N with the induced orientation from N. Find a positive basis of $T_p \partial N$ for $p = (1, 0, 0)$.

(3) Let $M_1 \subset \mathbb{R}^{n_1}, M_2 \subset \mathbb{R}^{n_2}$ be orientable manifolds without boundary. Prove that $M_1 \times M_2 \subset \mathbb{R}^{n_1+n_2}$ is orientable.

(4) Let $M^k \subset \mathbb{R}^n$ be a C^∞ manifold with boundary. Prove that for any $p \in \partial M$ there exists an open set $U \subset \partial M$ containing p on which we can construct a C^∞ unit vector filed N tangent to M such that $N(x) \perp T_x \partial M$ for any $x \in U$.

Hint: Take a local parametrization $f\colon V \to U$ where $V \subset H^k$, $U \subset M$ and look at the vector fields $df_x(e_1), \ldots, df_x(e_k)$. Apply Gramm-Shmidt to those vector fields.

(5) Let $M = \{x^2/9 + y^2/4 + z^2 \leq 3\}$ in R^3. Consider the induced orientation on ∂M and find a positive basis of $T_p \partial M$ at $p = (3, -2, 1)$.

(6) Let $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with the orientation induced from the ball $\{x^2 + y^2 + z^2 \leq 1\}$. Let $\omega = z dx \wedge dy$. Compute $\int_M \omega$.