(1) Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold. Let ω be an l-form on M. recall that ω is called smooth if it can be extended to a smooth form on an open set containing M.

a) Prove that ω is smooth if and only if it's locally smooth. Here a form on M is locally smooth if for every $p \in M$ there exists open subset $U \subset \mathbb{R}^n$ containing p such that $\omega|_{M\cap U}$ is smooth.
Hint: use partition of unity.

b) Prove that ω is smooth if and only if for any smooth tangent fields $V_1(x), \dots V_l(x)$ on M the function $\omega(V_1(x), \dots V_l(x))$ is smooth in x.
Hint: For the if direction: by a) it's enough to argue locally. Extend local coordinates on M to a local diffeomorphism between open sets in $\mathbb{R}^n$, look at the form in those local coordinates and extend it there.

(2) Let $U \subset \mathbb{R}^k, V \subset \mathbb{R}^n$ be open where $n \geq k$. Let $\omega = dy_I$ be a k-form on V where $I = (i_1 < i_2 < \dots < i_k)$. Let $f = (f_1, f_2, \dots, f_n)\colon U \to V$ be smooth. Let $f_I = (f_{i_1}, f_{i_2}, \dots, f_{i_k})\colon \mathbb{R}^k \to \mathbb{R}^k$
Prove that $f^*(dy_I)(x) = \det[df_x]dx_1 \wedge dx_2 \dots \wedge dx_k$.

(3) Let $f\colon R^3 \to R^3$ be given by $f(x,y,z) = (y\sin(z), xe^z, 1+y^2)$. Let $\omega = zdx \wedge dy$. Compute $df^*(\omega)$ and $f^*(d\omega)$ and verify that they are equal .

(4) Prove that every closed C^∞ 1-form on $\mathbb{R}^2$ is exact.
Hint: Let $\omega = P(x,y)dx + Q(x,y)dy$ with $d\omega = 0$. We want to find a function $F(x,y)$ such that $\omega = dF$, i.e. $P = \frac{\partial F}{\partial x}$ and $Q = \frac{\partial F}{\partial y}$.
Define $F(x,y) = \int_0^x P(x,0)dx + \int_0^y Q(x,y)dy$. Use that $d\omega = 0$ to show that $dF = \omega$.

(5) A subset $X \subset \mathbb{R}^n$ is called path connected if for any points $p, q \in X$ there exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p, \gamma(1) = q$. Let $U\colon \mathbb{R}^n$ be an open path connected set and $f\colon U \to V$ be a C^1 diffeomorphism onto an open set $V \subset \mathbb{R}^n$.
Prove that $\det[df_x] > 0$ for all $x \in U$ or $\det[df_x] < 0$ for all $x \in U$.

(6) Let $\sigma\colon (0,1)^2 \to R^3$ be given by $\sigma(x,y) = (xy, 2x+y, y^2)$. Let ω be a 2-form on R^3 given by $x_1 dx_2 \wedge dx_3 + x_2^2 dx_1 \wedge dx_3$.
Find $\int_\sigma \omega$.

(7) Let $U \subset \mathbb{R}_n$ be open and $w \subset \Omega^1(U)$ be exact. Let $p, q \in U$ be fixed and let $\gamma\colon [0,1] \to U$ be C^1 such that $\gamma(0) = p, \gamma(1) = q$.
Prove that $\int_\gamma \omega$ is independent of γ.

(8) Let $\omega = \frac{xdy - ydx}{x^2+y^2}$ be a 1-form on $\mathbb{R}^2 \backslash (0,0)$

(a) Verify that ω is closed.
(b) Prove that ω is not exact.

Hint: Use the previous problem.

Extra Credit: John Nash's Problem.
Is it true that every closed 1-form on $R^3\backslash\{(0,0,0)\}$ is exact?