(1) Let $M \subset \mathbb{R}^n$ be a C^r manifold and let $f, g\colon M \to \mathbb{R}^m$ be C^r maps such that $f|_M = g|_M$. Let $p \in M$.

Prove that $df_p|_{T_pM} = dg_p|_{T_pM}$.

(2) Let $M \subset \mathbb{R}^n$ be a smooth (i.e. C^∞) manifold. Prove that there exists a smooth tangent vector field defined on M which is not identically zero on M. *Hint:* Use partition of unity to glue together locally defined tangent vector fields.

(3) Let V be a smooth vector field on $\mathbb{R}^n$. Let $M \subset \mathbb{R}^n$ be a smooth manifold. Let for $p \in M$ let $V^t(p)$ be the result of the orthogonal projection of $V(p)$ to T_pM.

Prove that V^t is smooth.

Hint: use that one can construct a local family of smooth orthonormal vector fields tangent to M.

(4) Let V be a n-dimensional vector space and let $\langle \cdot, \cdot \rangle$ be a scalar product on V and let μ be an orientation on V.

Prove that there exists a unique alternating n-tensor $\omega \in \mathcal{A}^n(V)$ such that $\omega(e_1, \dots, e_n) = 1$ for any positively oriented orthonormal basis $e_1, \dots, e_n$ of V.

(5) Let Let $M \subset \mathbb{R}^3$ be given by $\{x^2 + y^2 - 5z^2 = 0\} \cap \{2x - y + z = 1\}$. Prove that M is a manifold and find T_pM for $p = (1, 2, 1)$.

(6) Prove the theorem stated in class: A l-tensor field T on a manifold $M \subset \mathbb{R}^k$ is smooth if an only if for any coordinate parameterization ϕ coming from the definition of a manifold we have that $T(\phi_*(e_{i_1})(p), \dots \phi_*(e_{i_l})(p))$ is smooth as a function of p for any $I = (i_1, \dots i_l)$.