(1) Let \(M \subset \mathbb{R}^n\) be a \(C^r\) manifold and let \(f, g: M \to \mathbb{R}^m\) be \(C^r\) maps such that \(f|_M = g|_M\). Let \(p \in M\).
Prove that \(df|_{T_pM} = dg|_{T_pM}\).

(2) Let \(M \subset \mathbb{R}^n\) be a smooth (i.e. \(C^\infty\)) manifold. Prove that there exists a smooth tangent vector field defined on \(M\) which is not identically zero on \(M\). *Hint:* Use partition of unity to glue together locally defined tangent vector fields.

(3) Let \(V\) be a smooth vector field on \(\mathbb{R}^n\). Let \(M \subset \mathbb{R}^n\) be a smooth manifold. Let for \(p \in M\) let \(V^t(p)\) be the result of the orthogonal projection of \(V(p)\) to \(T_pM\).
Prove that \(V^t\) is smooth.
*Hint:* use that one can construct a local family of smooth orthonormal vector fields tangent to \(M\).

(4) Let \(V\) be a \(n\)-dimensional vector space and let \(\langle \cdot, \cdot \rangle\) be a scalar product on \(V\) and let \(\mu\) be an orientation on \(V\).
Prove that there exists a unique alternating \(n\)-tensor \(\omega \in A^n(V)\) such that \(\omega(e_1, \ldots, e_n) = 1\) for any positively oriented orthonormal basis \(e_1, \ldots, e_n\) of \(V\).

(5) Let \(M \subset \mathbb{R}^3\) be given by \(\{x^2 + y^2 - 5z^2 = 0\} \cap \{2x - y + z = 1\}\).
Prove that \(M\) is a manifold and find \(T_pM\) for \(p = (1, 2, 1)\).

(6) Prove the theorem stated in class: A \(l\)-tensor field \(T\) on a manifold \(M \subset \mathbb{R}^k\) is smooth if and only if for any coordinate parameterization \(\phi\) coming from the definition of a manifold we have that \(T(\phi_*(e_{i_1})(p), \ldots, \phi_*(e_{i_l})(p))\) is smooth as a function of \(p\) for any \(I = (i_1, \ldots, i_l)\).