(1) Let X be a metric space and let $f\colon X \to \mathbb{R}$ be a function. Define support of f (which will denote as $supp(f)$) as the closure of the set of points $x \in X$ where $f(x) \neq 0$.

Prove that $X \backslash supp(f) = \{x \in X$ such that there exists $\epsilon > 0$ such that f is identically zero on $B_\epsilon(f)\}$.

(2) Consider the modified Cantor set S on $[0,1]$ constructed as follows. Let S_1 be obtained from $[0,1]$ by removing the open interval $(\frac{1}{2} - \frac{1}{2\cdot 5}, \frac{1}{2} + \frac{1}{2\cdot 5})$ of length $\frac{1}{5}$. Note that S_1 is a union of two closed intervals $I_1 = [0, \frac{1}{2} - \frac{1}{2\cdot 5}]$ and $I_2 = [\frac{1}{2} + \frac{1}{2\cdot 5}, 1]$. Let S_2 be obtained from S_1 by further removing the "middle" open intervals of length $\frac{1}{5^2}$ from I_1 and I_2 etc. Let $S = \cap_{i=1}^{\infty} S_i$ be the Cantor set. Let $U = [0,1] \backslash S$

Prove that $\int_U^{ext} 1$ exists and compute it.

(3) Determine if $\int_{(1,\infty)}^{ext} \frac{\sin x}{x} dx$ exists.

(4) Let $U_1, U_2 \subset \mathbb{R}^n$ be open. Let $U = U_1 \cup U_2, V = U_1 \cap U_2$. Let $f\colon U \to \mathbb{R}$ be continuous almost everywhere and locally bounded. Suppose $\int_{U_1}^{ext} f$ and $\int_{U_2}^{ext} f$ exist. Prove that $\int_U^{ext} f$ and $\int_V^{ext} f$ also exist and

$$\int_U^{ext} f = \int_{U_1}^{ext} f + \int_{U_2}^{ext} f - \int_V^{ext} f$$

(5) Let $U = \{(x,y,z) \in \mathbb{R}^3|$ such that $z > 0, x^2 + y^2 + z^2 < 1\}$.

Find $\int_U^{ext} z$ using spherical change of variables.

(6) Let $U = \{(x,y) \in R^2 | x^2 + y^2 < 1\}$. Let $f(x,y) = e^{x^2+y^2}$. Find $\int_U^{ext} f$.

(7) Finish the proof of the following theorem from class:

Let $f\colon U \to V$ be a diffeomorphism between open subsets of R^n. Let $C \subset U$ be a compact subset.

Prove that $f(bd(C)) = bd(f(C))$.

Extra Credit Problem (to be written up and submitted separately)

Give an example of a diffeomorphism between open sets in R^n which is not C^1.

Hint: Look at the map $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 3x + x^2 \sin(1/x) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$