MAT 257Y　　Solutions to Practice Final 2

1. Let $A \subset \mathbb{R}^n$ be a rectangle. Let $f\colon A \to \mathbb{R}$ be integrable. Let

$$f_+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases}$$

Prove that f_+ is also integrable on A.

Solution

Since f is integrable there exists a set of measure zero $S \subset A$ such that f continuous at every $x \in A \backslash S$. Also f is bounded on A, that is $|f(x)| \leq M$ for all $x \in A$ for some $M > 0$.

We have $f_+(x) = \frac{1}{2}(f(x) + |f(x)|)$ is also continuous on $x \in A \backslash S$ since $g(y) = |y|$ is continuous everywhere.

Clearly $|f_+(x)| \leq M$ for any $x \in A$. Therefore, by the criterion of integrability, f_+ is integrable on A.

2. Mark True or False. **If true, give a proof. If false, give a counterexample.**
   (a) Let $S \subset \mathbb{R}^n$. If $bd(S)$ is rectifiable then S is rectifiable.
   (b) Let $A, B \subset \mathbb{R}^n$. Then $bd(A \cap B) = bd(A) \cap bd(B)$;
   (c) Let $A \subset \mathbb{R}^n$. Then $int(intA) = int(A)$
   (d) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If $A \subset \mathbb{R}^n$ is open then $f(A)$ is open.

Solution

(a) **False**. For example, take $S = [0, \infty) \subset \mathbb{R}$. Then $bd(S) = \{0\}$ is rectifiable but S is not as it's not bounded.

(b) **False**. For example, take $A = \mathbb{Q}$ and $B = \mathbb{R}\backslash\mathbb{Q}$. Then $bd(A) = bd(B) = \mathbb{R}$ so that $bd(A) \cap bd(B) = \mathbb{R}$. But $A \cap B = \emptyset$ and hence $bd(A \cap B) = \emptyset$.

(c) **True**. $int(A)$ is open and $int(U) = U$ for any open set U.

(d) **False**. Let $f(x) \equiv 0$ and $A = \mathbb{R}^n$. Then A is open but $f(A) = \{0\}$ is not.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 \text{ if } (x,y) = (0,0) \\ \frac{2x^3+xy^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \end{cases}$$

(a) Show that the partial derivatives $D_1f(0,0), D_2f(0,0)$ exist and compute them.

(b) Is f differentiable at $(0,0)$? If yes, find $df_{(0,0)}$. If no, explain why not.
Hint: use part a).

Solution

(a) We compute $f(x,0) = \frac{2x^3}{x^2} = 2x$. Note that this formula remains true for $x = 0$ as $f(0,0) = 0$ and $2 \cdot 0 = 0$. Therefore, $D_1f(0,0) = 2$. Similarly, $f(0,y) = 0$ so that $D_2f(0,0) = 0$.

(b) We claim that f is not differentiable at $(0,0)$?If it were differentiable then the differential would be given by $B(x,y) = 2x$ by part a). However, we compute for $v = (1,1)$ that $D_vF(0,0) = \lim_{t\to 0} \frac{2t^3+t^3}{2t^3} = \frac{3}{2} \neq B(1,1) = 2$.
Therefore f is not differentiable at $(0,0)$.

4. Let $F(x,y) = \int_x^y \sqrt{e^{tx}+3y}dt$. Let $c = F(0,1)$.
Show that near $(0,1)$ the level set $F(x,y) = c$ can be written as $y = g(x)$ for some differentiable function g and compute $g'(0)$.

Solution

First we evaluate $\frac{\partial F(x,y)}{\partial x} = -\sqrt{e^{x^2}+3y} + \int_x^y \frac{te^{tx}}{2\sqrt{e^{tx}+3y}}dt$ and $\frac{\partial F(x,y)}{\partial y} = \sqrt{e^{xy}+3y} + \int_x^y \frac{3}{2\sqrt{e^{tx}+3y}}dt$. Plugging in $x=0, y=1$ we get $\frac{\partial F(1,0)}{\partial x} = -\sqrt{e^{0^2}+3} + \int_0^1 \frac{te^0}{2\sqrt{e^0+3}}dt = -2 + \int_0^1 \frac{t}{4}dt = -2 + \frac{1}{8} = -\frac{7}{8}$ and $\frac{\partial F(1,0)}{\partial y} = \sqrt{e^0+3} +$

$\int_0^1 \frac{3}{2\sqrt{e^0+3}}dt = 2 + \int_0^1 \frac{3}{4}dt = 2 + \frac{3}{4} = \frac{11}{4}$. Since $\frac{\partial F(1,0)}{\partial y} \neq 0$, by the Implicit Function theorem we conclude that near $(0,1)$ the level set $F(x,y) = c$ can be written as a graph of a differentiable function $y = g(x)$ and

$$g'(0) = -\frac{\frac{\partial F(1,0)}{\partial x}}{\frac{\partial F(1,0)}{\partial y}} = -\frac{-\frac{7}{8}}{\frac{11}{4}} = \frac{7}{22}$$

5. Let η be an alternating k-tensor on a vector space V. Let $v_1, \dots v_k \in V$ be linearly dependent.

   Show that $\eta(v_1, \dots, v_k) = 0$.

Solution

WLOG we can assume that v_1 is a linear combination of $v_2, \dots, v_k$, that is $v_1 = \sum_{i=2}^k \lambda_i v_i$. Therefore $\eta(v_1, \dots, v_k) = \eta(\sum_{i=2}^k \lambda_i v_i, v_2, \dots, v_k) = \sum_{i=2}^k \lambda_i \eta(v_i, \dots, v_i, \dots, v_k) = 0$ since η is alternating.

6. Let $M^3 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $1 \leq x^2+y^2+z^2 \leq 4\}$ with the orientation induced from $\mathbb{R}^3$.

   Let $p = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Find a positive basis of $T_p\partial M$ with respect to the orientation of ∂M induced from M.

Solution

It's easy to see that that outward unit normal to M at p is $n = -(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the tangent space T_pM is given by $x + y + z = 0$. Let $u_1 = (1,-1,0), u_2 = (1,0,-1)$. This is obviously a basis of T_pM. To determine if this basis if positive we compute the sign of the

$$\det(n, u_1, u_2) = \det\begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} =$$

$$= -\frac{1}{\sqrt{3}} \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = -\sqrt{3} < 0$$

Thus, this basis is negative. Therefore the basis $-u_1 = (-1,1,0), u_2 = (1,0,-1)$ is positive.

7. Let (X,d) be a metric space.
   (a) Let $p \in X$ be any point. Prove that $\{p\}$ is a closed subset of X.
   (b) Let $C \subset X$ be compact. Prove that C is closed. **You are not allowed to use any theorems about compact sets in the proof.**

Solution

(a) It's enough to show that $X\backslash\{p\}$ is open. Let $q \in X\backslash\{p\}$ Then $p \neq q$. Let $\varepsilon = \frac{d(p,q)}{2}$. Then $p \notin B_\varepsilon(q)$, i.e. $B_\varepsilon(q) \subset X\backslash\{p\}$.

(b) It's enough to show that $X\backslash C$ is open. Let $p \in X\backslash C$. Let $U_n = \{x \in X|$ such that $d(x,p) > \frac{1}{n}\}$. Then U_n is open and $\cup_{n=1}^{\infty} U_n = X\backslash\{p\} \supset C$. Therefore we can choose a finite cover of C out of this open cover. as the sets U_n are nested this means that $C \subset U_m$ for some m which means that $B(p, \frac{1}{m}) \subset X\backslash C$.

8. Let $U = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 > 1\}$. Let $f(x,y) = \frac{y}{x^2+y^2}$.
   Determine if $\int_U^{ext} f$ exists and if it does compute it.

Solution

Let $U_n = \{1 < x^2+y^2, n^2\}$. Then U_n form an open exhaustion of U so that $\int_U^{ext} f$ exists iff $\lim_{n\to\infty} \int_{U_n}^{ext} |f|$ exists. Let $V_n = U_n \backslash [0\infty) \times \{0\}$. Then f is integrable on U_n and we have $\int_{U_n}^{ext} |f| = \int\int_{V_n}^{ext} |f|$. By making polar coordinates change of variables we get

$$\int\int_{V_n}^{ext} |f| = \int_0^n \left(\int_0^{2\pi} \frac{|r\sin\theta|}{r^2} r d\theta\right) dr = 2\int_0^n \int_0^\pi \sin\theta d\theta dr = 4n$$

Therefore, $\lim_{n\to\infty} \int_{U_n}^{ext} |f|$ does not exists and hence $\int_U^{ext} f$ does not exist.

9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(s,t) = (st, s+2t)$ and let $\omega = \sin x dy$. Compute $f^*(d\omega)$ and $d(f^*\omega)$ and verify that they are equal.

Solution

We compute $d\omega = \cos x dx \wedge dy$ and $f^*(d\omega) = \cos(st) d(st) \wedge d(s+2t) = \cos(st)(sdt + tds) \wedge (ds + 2dt) = \cos(st)(2t - s) ds \wedge dt$.

Next, $f^*(\omega) = \sin(st) d(s+2t) = \sin(st) ds + 2\sin(st) dt$ and $df^*(\omega) = d\sin(st) \wedge ds + 2d\sin(st) \wedge dt = (\cos(st) sdt + \cos(st) tds \wedge ds + 2(\cos(st) sdt + \cos(st) tds \wedge dt) = -\cos(st) sds \wedge dt + 2t\cos(st) ds \wedge dt = \cos(st)(2t-s) ds \wedge dt$

10. Let $a, b > 0$ and Let $M \subset \mathbb{R}^2$ be the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ with the orientation induced by the standard orientation on $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$.

Find $\int_M (\cos x) y dx + (x + \sin(x)) dy$.

Solution

Let $\omega = (\cos x) y dx + (x + \sin(x)) dy$.

Note that $M = \partial N$ where $N = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$ taken with the standard orientation coming from $\mathbb{R}^2$. By Stokes's Theorem this gives $\int_M \omega = \int_N d\omega$. We compute $d\omega = (\cos x) dy \wedge dx + (1 + \cos x) dx \wedge dy = dx \wedge dy$. Thus $\int_N d\omega = \int_N 1$. Using the change of avriables $x = au, y = bv$ we get $\int_N 1 = \int_{\{u^2+v^2 \leq 1\}} ab = \pi ab$.