MAT 257Y Solutions to Practice Term Test 3

(1) Let $v_1, \dots, v_k \in \mathbb{R}^n$ where $n \geq k$. Prove that $\mathrm{vol}_k P(v_1, \dots, v_k) = 0$ if and only if $v_1, \dots, v_k$ are linearly dependent.

Solution

Let A be the $n \times k$ matrix with columns $v_1, \dots, v_k$. Then by definition, $\mathrm{vol}_k P(v_1, \dots, v_k) = \sqrt{\det A^t \cdot A}$. If $v_1, \dots, v_k$ are linearly dependent then $rank(A) < k$ and hence $rank(A^t \cdot A) \leq rank A < k$ which means that $\det A^t \cdot A = 0$ and hence $\mathrm{vol}_k P(v_1, \dots, v_k) = 0$.

Conversely, if $\mathrm{vol}_k P(v_1, \dots, v_k) = 0$ then $\det A^t \cdot A = 0$.

Therefore there exists $x = (x_1, \dots, v_k) \in R^k$ such that $x \neq 0$ and $A^t \cdot Ax = 0$. Hence, $0 = \langle A^t Ax, x \rangle = \langle Ax, Ax \rangle = ||Ax||^2$. Therefore $Ax = 0$. But $Ax = x_1 v_1 + \ldots + x_k v_k$ and hence $v_1, \dots, v_k$ are linearly dependent.

(2) Let T be a k-tensor on $\mathbb{R}^n$. Prove that T is C^∞ as a map $\mathbb{R}^{nk} \to \mathbb{R}$.

Solution

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Let $v_1, \dots, v_k \in \mathbb{R}^n$. we can write them in coordinates $v_i = \sum_j x_i^j e_j$

Then $T(v_1, \dots, v_k) = T(\sum_{j_1} x_1^{j_1} e_{j_1}, \dots, \sum_{j_k} x_k^{j_k} e_{j_k}) = \sum_{j_1, \dots, j_k} x_1^{j_1} \cdot \ldots \cdot x_k^{j_k} T(e_{j_1}, \dots, e_{j_k})$. This is a polynomial in x_j^i's and hence is C^∞.

(3) Let M be a union of x and y axis in $\mathbb{R}^2$. Prove that M is not a C^1 manifold.

Solution

Suppose M is a C^1 manifold. Then there exists an open neighborhood $U \subset \mathbb{R}^2$ of the origin and a C^1 map $f\colon U \to \mathbb{R}$ such that $c = f(0,0)$ is a regular value and $M \cap U = f^{-1}(c)$. But then $f(x, 0) = 0$ on U and hence $\frac{\partial f}{\partial x}(0,0) = 0$. Similarly, $f(0, y) = 0$ on

U and hence $\frac{\partial f}{\partial y}(0,0) = 0$. Hence $df_{(0,0)} = 0$ which means that $c = f(0,0)$ is not a regular value. This is a contradiction and hence M is not a manifold.

(4) Prove that $S^2_+ = \{(x,y,z) \in \mathbb{R}^3 |$ such that $x^2 + y^2 + z^2 = 1, z \geq 0\}$ is a manifold with boundary.

Solution

Consider the following parametrization $f(\theta, \phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ where $(\theta,\phi) \in U_a = \{a < \theta < a + 2\pi, 0 \leq \phi < \pi/2\}$ for a fixed $a \in \mathbb{R}$. Note that $U_a \subset H^2$ is open in H^2.

Also, ϕ is C^∞, 1-1 with continuous inverse, and $[df] = [\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}]$ has rank=2 everywhere. Indeed, we compute

$\frac{\partial f}{\partial \theta} = (-\cos\phi\sin\theta, \cos\phi\cos\theta, 0)$ and

$\frac{\partial f}{\partial \phi} = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi)$

We compute $\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi} = (\cos^2\phi\cos\theta, \cos^2\phi\sin\theta, \cos\phi\sin\phi)$ and hence $|\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}|^2 = \cos^4\phi\cos^2\theta + \cos^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi = \cos^4\phi + \cos^2\phi\sin^2\phi = \cos^2\phi \neq 0$ for $0 \leq \phi < \pi/2$. This means that $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial \phi}$ are linearly independent and hence $[df]$ has rank=2. Therefore f satisfies the definition of a paramterization in a definition of a manifold with boundary. By varying a we can cover all of S^2_+ by images of such parametrizations with the exception of the north pole $p = (0,0,1)$. However, near this point S^2_+ is given by the graph of a C^∞ function $z = \sqrt{1 - x^2 - y^2}$ and therefore it admits a parametrization near p also.

(5) Let $c\colon [0,1] \to (\mathbb{R}^n)^n$ be continuous. Suppose that $c^1(t), \ldots, c^n(t)$ is a basis of $\mathbb{R}^n$ for any t.

Prove that $(c^1(0), \ldots, c^n(0))$ and $(c^1(1), \ldots, c^n(1))$ have the same orientation.

Solution

Let $f(t) = \det[c^1(t), \ldots, c^n(t)]$. Then $f(t)$ is continuous and never zero. therefore $f(t) > 0$ for all t or $f(t) < 0$ for all t by the intermediate value theorem. In either case $f(1)/f(0) > 0$. Let A be the transition matrix from $(c^1(0), \ldots, c^n(0))$ to $(c^1(1), \ldots, c^n(1))$. then $A = [c^1(0), \ldots, c^n(0)]^{-1}[c^1(1), \ldots, c^n(1)]$. hence $\det(A) = f(1)/f(0) > 0$ which means that $(c^1(0), \ldots, c^n(0))$ and $(c^1(1), \ldots, c^n(1))$ have the same orientation.

(6) Let C be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,2), (-1,3)$ Compute $\int_C x + y$.

Solution

Let's make a change of variable

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

or $x = u - v, y = 2u + 3v$.

We have that $\det \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = 5$. Therefore,

$\int_C x + y = \int_U 5((u - v) + (2u + 3v))$ where $U = \{(u,v) | u > 0, v > 0, u + v < 1\}$. Therefore using Fubini's theorem we compute

$$\int_U 5((u-v) + (2u+3v)) = \int_0^1 \int_0^{1-u} 5(3u + 2v) dv du =$$

$$= 5 \int_0^1 (3uv + v^2)|_0^{1-u} du = 5 \int_0^1 3u(1-u) + (1-u)^2 du =$$

$$= 5 \int_0^1 -2u^2 + u + 1 du = 5(-2/3u^3 + u^2/2 + u)|_0^1 = 25/6$$

(7) Let e_1, e_2 be a basis of a vector space V of dimension 2. Let $T \in \mathcal{L}^2(V)$ be given by $e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$.

Prove that T can not be written as $S \otimes U$ with $S, U \in \mathcal{L}^1(V)$.

Solution

Suppose $e_1^* \otimes e_1^* + e_2^* \otimes e_2^* = S \otimes U$ for some $S = ae_1^* + be_2^*$, $U = ce_1^* + de_2^*$. Then $S \otimes U = (ae_1^* + be_2^*) \otimes (ce_1^* + de_2^*) = ace_1^* \otimes e_1^* + bce_2^* \otimes e_1^* + ade_2^* \otimes e_1^* + bde_2^* \otimes e_2^* = e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$. This means that $ac = 1, bc = 0, ad = 0, bd = 1$. It's easy to see that this system has no solutions. for example, $abcd = (bc)(ad) = 0 \cdot 0 = 0$ and on the other hand, $abcd = (ac)(bd) = 1 \cdot 1 = 1$. This is a contradiction.

(8) Let $U \subset R^n$ be open. Let $f, g\colon U \to R$ be continuous and $|f| \leq g$. Suppose $\int_U^{ext} g$ exists.

Prove that $\int_U^{ext} f$ also exists.

Solution

Let ϕ_i be a partition of unity on U. Then by definition of extended integral, $\sum_{i=1}^{\infty} \int_U |g|\phi_i < \infty$

Therefore

$\sum_{i=1}^{\infty} \int_U |f|\phi_i \leq \sum_{i=1}^{\infty} \int_U |g|\phi_i < \infty$ and hence $\int_U^{ext} f$ exists by the definition.