MAT 257Y Solutions to Practice Final

(1) Let $A \subset R^n$ be a rectangle and let $f\colon A \to R$ be bounded. Let P_1, P_2 be two partitions of A . Prove that $L(f, P_1) \leq U(f, P_2)$.

Solution

The statement is obvious if $P_1 = P_2$. In general, let P' be a common refinement of P_1 and P_2. Then $L(f, P_1) \leq L(f, P')$. Indeed, for any rectangle Q' of P' contained in a rectangle Q of P_1 we have that $m(f, Q) \leq m(f, Q')$. therefore

$$L(f, P') = \sum_{Q' \in P'} m(f, Q')\mathrm{vol}Q' = \sum_{Q \in P} \sum_{Q' \subset Q} m(f, Q')\mathrm{vol}Q' \geq$$

$$\sum_{Q \in P} \sum_{Q' \subset Q} m(f, Q)\mathrm{vol}Q' = \sum_{Q \in P} m(f, Q) \sum_{Q' \subset Q} \mathrm{vol}Q' =$$

$$\sum_{Q \in P} m(f, Q)\mathrm{vol}Q = L(f, P_1)$$

Thus $L(f, P_1) \leq L(f, P')$ and similarly $U(f, P') \leq U(f, P_2)$. This finally gives

$$L(f, P_1) \leq L(f, P') \leq U(f, P') \leq U(f, P_2)$$

(2) let $M = \{(x, y) \in R^2|$ such that $x^2 + y^2 = 1$. let $f\colon M \to R$ be given by $f(x, y) = x^2 + y$. Find the minimum and the maximum of f on M.

Solution

Let $g(x, y) = x^2 + y^2$. By the Lagrange multiplier method extremum points of f on M can only occur when $\nabla f = \lambda \nabla g$. We have $\nabla g(x, y) = (2x, 2y)$ and

$\nabla f(x,y) = (2x,1)$. We need to solve

$$\begin{cases} 2x = \lambda 2x \\ 2y = \lambda \\ x^2+y^2 = 1 \end{cases}$$

If $x \neq 0$ the first equation give $\lambda = 1$. hence $y = 1/2$ and $x = \pm\frac{\sqrt{3}}{2}$.

If $x = 0$ then $y = \pm 1$. Thus we have four possible points we need to test $(0,1), (0,-1), (\frac{\sqrt{3}}{2}, 1/2)$ and $(-\frac{\sqrt{3}}{2}, 1/2)$.

Computing f at these points we get $f(0,1) = 1, f(0,-1) = -1, f(\frac{\sqrt{3}}{2}, 1/2) = f(-\frac{\sqrt{3}}{2}, 1/2) = 3/4 + 1/2 = 5/4$. Thus the maximum of f on M os $5/4$ and the minimum is -1.

(3) Let $T\colon R^{2n} = R^n \times R^n \to R$ be a 2-tensor on R^n. Show that T is differentiable at $(0,0)$ and compute $dT(0,0)$.

Solution

Let $x = (x_1, \ldots x_n)$ be the coordinates on the first copy of R^n and $y = (y_1, \ldots, y_n)$ on the second. Then by multilinearity we have that $T(x,y) = \sum_{ij} T_{ij} x_i y_j$. This function is a polynomial and hence is differentiable. It is also obvious to check that its partial derivatives at $(0,0)$ are all zero. therefore $dT(0,0) = 0$.

(4) Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $R^3 \backslash (0,0,0)$.

Verify that ω is closed.

Hint: One way to simplify the computation is to write $\omega = f \cdot \tilde{\omega}$ where $f = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ and $\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx$.

Solution

We have $d\omega = df \wedge \tilde{\omega} + (-1)^0 f d\tilde{\omega}$

$$df \wedge \tilde{\omega} = -\frac{3}{2}\frac{1}{(x^2+y^2+z^2)^{5/2}}(2xdx+2ydy+2zdz)\wedge(xdy\wedge dz+ydz\wedge dx+zdx)$$

$$= -\frac{3(x^2+y^2+z^2)dx\wedge dy\wedge dz}{(x^2+y^2+z^2)^{5/2}} = -\frac{3dx\wedge dy\wedge dz}{(x^2+y^2+z^2)^{3/2}}$$

$$f d\tilde{\omega} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \cdot 3dx\wedge dy\wedge dz = \frac{3dx\wedge dy\wedge dz}{(x^2+y^2+z^2)^{3/2}}$$

Therefore $d\omega = df \wedge \tilde{\omega} + (-1)^0 f d\tilde{\omega} = 0$.

(5) Let $f\colon R^2 \to R^2$ be given by $f(x,y) = (e^{2y}, 2x+y)$ and let $\omega = x^2ydx + ydy$.

Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

Solution

$d\omega = (2xydx + x^2dy) \wedge dx + dy \wedge dy = -x^2dx \wedge dy$. $f^*(d\omega) = -(e^{2y})^2 de^{2y} \wedge (2dx+dy) = -e^{6y}dy \wedge (2dx + dy) = 2e^{6y}dx \wedge dy$.

On the other hand, $f^*(\omega) = (e^{2y})^2(2x+y)de^{2y} + (2x+y)(2dx+dy) = e^{6y}(2x+y)dy + (2x+y)\wedge(2dx + dy)$.

Finally, $d(f^*(\omega)) = d(e^{6y}(2x+y))\wedge dy + (2dx+dy)\wedge(2dx+dy) = (6e^{6y}(2x+y)dy + e^{6y}(2dx+dy)\wedge dy + 0 = 2e^{6y}dx \wedge dy$.

(6) Determine if $\int^{ext}_{0<x^2+y^2<1} \ln(x^2+y^2)$ exists and if it does compute it.

Solution

Let $U = 0 < x^2+y^2 < 1\backslash\{(0,1)\times 0\}$.

then $\int^{ext}_{0<x^2+y^2<1} \ln(x^2+y^2)$ exists iff $\int^{ext}_U \ln(x^2+y^2)$ exists and if they both exist they are equal. For the second integral make a polar change of variables $x = r\cos\theta, y = r\sin\theta$ where $0 < r < 1, 0 < \theta < 2\pi$. Then $\int^{ext}_U \ln(x^2+y^2) = \int_0^{2\pi}(\int_0^1 r\ln r^2 dr)d\theta =$

$4\pi\int_0^1 r\ln r dr = 4\pi\int_0^1 \ln r d(r^2/2) = 4\pi(\frac{r^2\ln r}{2}|_0^1 - \int_0^1 \frac{r^2}{2} d\ln r) = 4\pi(0-\int_0^1 r/2dr) = -4\pi r^3/6|_0^1 = -2\pi/3$. Here we used the fact that $\lim_{r\to 0+} r^2\ln r = 0$.

Thus $\int_{0<x^2+y^2<1}^{ext} \ln(x^2+y^2) = -2\pi/3$

(7) Let U, V be open in R^n. Let $f\colon R^n \to R$ be a continuous nonnegative function such that $\int_U^{ext} f$ and $\int_V^{ext} f$ exist.

Prove that $\int_{U\cup V}^{ext} f$ exists.

Hint: use compact exhaustions of U and V to construct a compact exhaustion of $U\cup V$.

Solution

let K_i be a compact exhaustion by measurable sets of U and C_i be a compact exhaustion by measurable sets of V. then we have that $\int_{K_i} f$ is increasing and $\lim_{i\to\infty}\int_{K_i} f = \int_U^{ext} f$. Similarly, $\lim_{i\to\infty}\int_{C_i} f = \int_V^{ext} f$.

Then it's easy to see that $K_i\cup C_i$ is a compact exhaustion by measurable sets of $U\cup V$. Since $f\geq 0$ we have that $\int_{C_i\cup K_i} f = \int_{C_i} f + \int_{K_i} f - \int_{C_i\cap K_i} f \leq \int_{C_i} f + \int_{K_i} f \leq \int_U^{ext} f + \int_V^{ext} f$. Therefore $\lim_{i\to\infty}\int_{K_i\cup C_i} f$ exists and hence so does $\int_{U\cup V}^{ext} f$.

(8) Let $F(x) = \int_{e^x}^{x^2} f(tx)dt$ where $f\colon R\to R$ is C^1.

Show that $F(x)$ is C^1 and find the formula for $F'(x)$.

Solution

Let $G(x,a,b) = \int_a^b f(tx)dt$. Then G is C^1 by a theorem from class and. $\frac{\partial G}{\partial x}(x,a,b) = \int_a^b \frac{d}{dx} f(tx)dt = \int_a^b tf'(tx)dt$. Also, $\frac{\partial G}{\partial b}(x,a,b) = f(bx)$ and $\frac{\partial G}{\partial a}(x,a,b) = -f(ax)$.

Then $F(x) = G(x, e^x, x^2)$ is C^1 by the chain rule and $F'(x) = \frac{\partial G}{\partial x}(x, e^x, x^2) + \frac{\partial G}{\partial a}(x, e^x, x^2) \cdot (e^x)' + \frac{\partial G}{\partial b}(x, e^x, x^2) \cdot (x^2)' = \int_{e^x}^{x^2} tf'(tx)dt - f(e^x x)e^x + f(x^3)2x$.

(9) Prove that a compact set is closed.

Solution

We will show that if $C \subset R^n$ is compact then $U = R^n \backslash C$ is open. let $p \in U$. let $U_i = \{x \in R^n |$ such that $d(x,p) > 1/i\}$. then V_i is open and $\cup_i V_i = R^n \backslash \{p\}$ covers C. By compactness C is covered by finitely many V_is and hence, $C \subset V_j$ for some j. This means that $B(p, 1/j) \subset U$. since $p \in U$ is arbitrary, this means that U is open and C is closed.

(10) Let $x(t_1, t_2) = t_1 \cos t_2, y(t_1, t_2) = t_1^2 + e^{t_1 t_2}$. Let $f(x, y)$ be a differentiable function $f\colon R^2 \to R$. Let $g(t_1, t_2) = f(x(t_1, t_2), y(t_1, t_2))$. Express $\frac{\partial g}{\partial t_1}(1, 0)$ and $\frac{\partial g}{\partial t_2}(1, 0)$ in terms of partial derivatives of f.

Solution

By the chain rule
$\frac{\partial g}{\partial t_1}(t_1, t_2) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\partial y}{\partial t_1} = \frac{\partial f}{\partial x}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2}) \cos t_2 + \frac{\partial f}{\partial y}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2})(2t_1 + t_2 e^{t_1 t_2})$.

Similarly, $\frac{\partial g}{\partial t_2}(t_1, t_2) = \frac{\partial f}{\partial x}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2})(-t_1 \sin t_2) + \frac{\partial f}{\partial y}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2})(t_1 e^{t_1 t_2})$.

(11) Mark true or false. Justify your answer.

Let A, B be any subsets of R^n.

(a) $br(A) \subset Lim(A)$

(b) $Lim(A) \subset A$

(c) $br(A \cap B) \subset br(A) \cap br(B)$.

Solution

(a) **False.** Example $A = \{p\}$. Then $brA = \{p\}$ and $Lim(A) = \emptyset$.

(b) **False.** Example $A = (0,1) \subset R$. Then $Lim(A) = [0,1]$ is not contained in A.

(c) **False.** Example $A = [0,2], B = [1,3]$. Then $A \cap B = [1,2]$ and $br(A \cap B) = \{1,2\}$. On the other hand, $br(A) \cap br(B) = \emptyset$.

(12) Let M^3 be a compact 3-manifold with boundary in R^3 and let n be the outward unit normal on ∂M. Let $F = (F_1, F_2, F_3)$ be a vector field on R^3. Prove that

$$\int_M divF = \int_{\partial M} \langle F, n \rangle$$

Hint: Convert the integral over ∂M to an integral of a form in R^3 and use Stokes' formula.

Solution

recall that $divF = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. Let $n = (n_1, n_2, n_3)$.
Let $\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$.
then $\int_{\partial M} \langle F, n \rangle dA = \int_{\partial M} \omega$. Indeed, $\langle F, n \rangle = F_1 n_1 + F_2 n_2 + F_3 n_3$. Recall that $n_1 dA = dy \wedge dz, n_2 dA = dz \wedge dz$ and $n_3 dA = dx \wedge dy$. Therefore,

$$\int_{\partial M} \langle F, n \rangle dA = \int_{\partial M} (F_1 n_1 + F_2 n_2 + F_3 n_3) dA =$$

$$\int_{\partial M} F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy = \int_{\partial M} \omega$$

By Stokes' formula this is equal to $\int_M d\omega = \int_M (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) dx \wedge dy \wedge dz = \int_M divF$.

(13) let $M^2 \subset R^3$ be the torus of revolution obtained by rotating the circle $(x-2)^2 + z^2 = 1$ in the xz plane around the yz axis. Consider the orientation on M induced by the outward normal field N where $N(3,0,0) = (1,0,0)$.

Find $\int_M x dy \wedge dz$

Solution

Let V be the solid obtained by rotating the disk $U = (x-2)^2 + z^2 \leq 1$ in the xz plane around the yz axis. then $M = \partial V$ and by Stokes' formula $\int_M xdy \wedge dz = \int_V d(xdy \wedge dz) = \int_V dx \wedge dy \wedge dz = \text{vol}V$. Recall that by a homework problem this is equal to $2\pi \int_U x$. Using polar coordinates change of variables $x = 2 + r\cos\theta, y = r\sin\theta$ we compute

$\int_U x = \int_0^{2\pi} \int_0^1 (2+r\cos\theta) r dr d\theta = \int_0^{2\pi} \int_0^1 (2r+r^2\cos\theta) dr d\theta = 2\pi$. Therefore $\int_M xdy \wedge dz = 4\pi^2$.

(14) Let $M \subset R^n$ be an oriented manifold.

Prove that $\text{vol}(M) = \int_M dA$ is positive.

Solution

Let U_i be a covering of M by orientation preserving coordinate patches $f_i \colon W_i \to M$ and let ϕ_i be the partition of unity subordinate to this covering. Note that $0 \leq \phi_i \leq 1$. Then $\int_M dA = \sum_i \int_M \phi_i dA = \sum_i \int_{W_i} f_i^*(\phi_i dA) = \sum_i \int_{W_i} (\phi_i \circ f_i) f_i^*(dA)$. Note that $\phi_i \circ f_i(x) \geq 0$ for any $x \in W_i$ and is positive at some point of W_i. We also have that $f_i^*(dA) = u(x) dx^1 \wedge \ldots \wedge dx^k$ where $u(x) = dA(f_{i*}e_1, \ldots, f_{i*}e_k) > 0$ since f_i is orientation preserving. Altogether the above means that $\int_{W_i} (\phi_i \circ f_i) f_i^*(dA) = \int_{W_i} g_i(x)$ where g_i is a continuos nonnegative function with compact support which is positive somewhere. therefore $\int_{W_i} g_i(x) > 0$ and hence $\int_M dA = \sum_i \int_M \phi_i dA > 0$.