Solutions to selected problems from homework 1

(1) The Fibonacci sequence is the sequence of numbers $F(0), F(1), \ldots$ defined by the following recurrence relations:

$F(0) = 1, F(1) = 1, F(n) = F(n-1) + F(n-2)$ for all $n > 1$.

For example, the first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, \ldots

Prove that

$$F(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

for all $n \geq 0$.

Solution

We prove the formula by induction on $n$.

First let’s check that the formula holds for $n = 0$ and $n = 1$.

For $n = 0$ we have $F(0) = 1$ and

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{1} \right] = \frac{1}{\sqrt{5}} \frac{1 + \sqrt{5} - (1 + \sqrt{5})}{2} = 1.$$

Thus the formula holds for $n = 0$.

For $n = 1$ we have $F(1) = 1$ and

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2} \right] = \frac{1}{\sqrt{5}} \frac{6 + 2\sqrt{5} - (6 - 2\sqrt{5})}{4} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Thus the formula also holds for $n = 1$.

Induction step. Suppose the formula holds for all $k = 0, 1, \ldots n$ for some $n \geq 1$. We need to show that

$$F(n+1) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \right]$$

Using that $\frac{1 + \sqrt{5}}{2}$ and $\frac{1 - \sqrt{5}}{2}$ satisfy the equation $1 + x = x^2$ we compute

By the induction assumption

$$F(n+1) = F(n) + F(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n} \right] + \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

$$= \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n} \left( 1 + \frac{1 + \sqrt{5}}{2} \right) - \left( \frac{1 - \sqrt{5}}{2} \right)^{n} \left( 1 + \frac{1 - \sqrt{5}}{2} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n} \left( \frac{1 + \sqrt{5}}{2} \right)^{2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n} \left( \frac{1 - \sqrt{5}}{2} \right)^{2} \right]$$

$$= \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \right]$$

(2) Using the method from class find the formula for the sum

$$1^3 + 2^3 + \ldots + n^3$$

Then prove the formula you’ve found by mathematical induction.

Solution
We’ve proved in class that \[ 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \text{ and } 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \]

Let’s find \( a_n \) such that \( a_1 + a_2 + \ldots + a_n = n^4 \). We have \[ n^4 = (a_1 + \ldots + a_{n-1}) + a_n = (n-1)^4 + a_n \text{ and hence } a_n = n^4 - (n-1)^4 = n^4 - (4n^3 + 6n^2 - 4n + 1) = 4n^3 - 6n^2 + 4n - 1. \]

Thus
\[
\begin{align*}
n^4 &= (4\cdot 1^3 - 6\cdot 1^2 + 4\cdot 1 - 1) + (4\cdot 2^3 - 6\cdot 2^2 + 4\cdot 2 - 1) + \ldots + (4\cdot n^3 - 6\cdot n^2 + 4\cdot n - 1) \\
&= 4(1^3 + 2^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4(1 + 2 + \ldots + n) - n \\
&= 4(1^3 + 2^3 + \ldots + n^3) - 6 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} - n \\
\end{align*}
\]
Therefore, \[ 4(1^3 + 2^3 + \ldots + n^3) = n^4 + n(n+1)(2n+1) - 2n(n+1) + n \]
\[ 1^3 + 2^3 + \ldots + n^3 = \frac{n^4 + n(n+1)(2n+1) - 2n(n+1) + n}{4} = \left(\frac{n(n+1)}{2}\right)^2 \]
Now that we have found the formula we can also prove it by induction.

First check that it holds for \( n = 1 \): \( 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 \). This verifies the base of induction.

Induction step: Suppose \( 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \) for some \( n \geq 1 \).

We need to show that \( 1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \).

We have \( 1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{(n(n+1)}{2}\right)^2 + (n + 1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \). \hfill \square.

(3) Find a mistake in the following "proof".

**Claim.** Any two natural numbers are equal.

We’ll prove the following statement by induction in \( n \): Any two natural numbers \( \leq n \) are equal.

We prove it by induction in \( n \).

a) The statement is trivially true for \( n = 1 \).

b) Suppose it’s true for \( n \geq 1 \). Let \( a, b \) be two natural numbers \( \leq n + 1 \). Then \( a - 1 \leq n \) and \( b - 1 \leq n \). Therefore, by the induction assumption

\[ a - 1 = b - 1 \]

Adding 1 to both sides of the above equality we get that \( a = b \). Thus the statement is true for \( n + 1 \). By the principle of mathematical induction this means that it’s true for all natural \( n \). \hfill \square.
Solution

The mistake is in step b) in the implication that since $a - 1 \leq n, b - 1 \leq n$ they must be equal by the induction assumption. The induction assumption is only valid for natural numbers which are all $\geq 1$. However $a - 1, b - 1$ need not be natural, one or both of them can be 0 which is not a natural number and therefore the induction assumption need not be applicable to $a - 1, b - 1$. For example this happens if $n = 1, a = 1, b = 2$. Then $a - 1 = 0, b - 1 = 1$.

(4) #14 from the book.

Solution

(a) We have that $F_n = 2^{2^n} + 1$ for $n = 0, 1, \ldots$ and we need to show that $F_0 \cdot \ldots \cdot F_{n-1} + 2 = F_n$ for $n \geq 1$. We do this by induction in $n$.

First we check the formula for $n = 1$. $F_0 = 2^{2^0} + 1 = 2^1 + 1 = 3, F_1 = 2^{2^1} + 1 = 5$ and $F_0 + 2 = 3 + 2 = 5 = F_1$. Thus the formula holds for $n = 1$.

Induction step. Suppose we know that $F_0 \cdot \ldots \cdot F_{n-1} + 2 = F_n$ for some $n \geq 1$. We need to show that $F_0 \cdot \ldots \cdot F_{n+1} = F_n + 2$.

We have $F_{n+1} = 2^{2^{n+1}} + 1 = 2^{2^n} + 1 = (2^{2^n})^2 + 1 = F_n^2 - 2F_n + 1 + 1 = F_n(F_n - 2) + 2$. By the induction assumption $F_n - 2 = F_0 \cdot \ldots \cdot F_{n-1}$ and therefore $F_{n+1} = (F_n - 2)F_n + 2 = F_0 \cdot \ldots \cdot F_{n-1} \cdot F_n + 2$.

(b) Let us first prove the following Claim $F_n, F_m$ have no common prime factors for $n \neq m$.

Let $m < n$. Suppose $p$ is a common prime factor for both $F_n, F_m$. Then $p|F_n$ and $p|F_0 \cdot \ldots \cdot F_{n-1}$ because the latter product contains $F_m$ as a factor. Thus we can write $F_n = ap, F_0 \cdot \ldots \cdot F_{n-1} = bp$ for some natural $a, b$. By part a) we know that $F_0 \ldots F_{n-1} + 2 = F_n$ which means that $bp + 2 = ap$. Hence $2 = ap - bp = p(a - b)$ and $p$ divides 2. Therefore $p = 2$ since that’s the only divisor of 2 bigger than 1. However $p$ can not be 2 since all Fermat numbers are odd. This is a contradiction and hence $F_n, F_m$ have no common prime divisors. This proves the Claim.

Next, for any $n \geq 1$ pick $p_n$ to be some prime divisor of $F_n$. This gives us a sequence of prime numbers $p_1, p_2, p_3, \ldots$. By the Claim above they are all distinct which implies that the set of prime numbers is infinite.