1. Vector fields

Definition 1.0.1. A vector field V on a manifold M is a map $V\colon M \to TM$ such that $\pi \circ V = id_M$ where $\pi\colon TM \to M$ is a canonical projection. I.e. V is a map $V\colon M \to TM$ such that $V(p) \in T_pM$ for any $p \in M$.

A vector field V is called smooth if it's smooth as a map $V\colon M \to TM$.

If V is a vector field on M^n and $x\colon U \to W$ is a local coordinate chart on M where U is open in M, W is open in $\mathbb{R}^n$ then $V|_U$ can be written as $V(p) = \Sigma_{i=1}^n v_i(p)\frac{\partial}{\partial x_i}|_p$ for some functions $v_i\colon U \to \mathbb{R}$.

Lemma 1.0.2. *Let V be a vector field on M^n then the following are equivalent*

1. *V is smooth;*
2. *$V(f)$ is smooth for any smooth $f\colon M \to \mathbb{R}$.*
3. *for any q there is a local coordinate chart $x\colon U \to W$ where $q \in U$, U is open in M, W is open in $\mathbb{R}^n$ such that $V(p) = \Sigma_{i=1}^n v_i(p)\frac{\partial}{\partial x_i}|_p$ and v_i are smooth functions on U.*

If V is a smooth vector field then V defines a map $D_V\colon C^\infty(M) \to C^\infty(M)$ given by $f \mapsto V(f)$. D_V satisfies $V(fg) = V(f)g + fV(g)$ for any $f, g \in C^\infty(M)$ i.e it's a derivation.

Let V be a smooth vector field on M. Let $0(M) \subset TM$ be the zero section. It is a submanifold of dimension n. It's a closed subset in TM.

Lemma 1.0.3. *Let $V = (V_1, \dots, V_n)$ be a vector field on an open set $U \subset \mathbb{R}^n$.*

Then V is transverse to the zero section iff for any $p \in U$ such that $V(p) = 0$ we have

$$\det\left(\frac{\partial V_i}{\partial x_j}(p)\right) \neq 0$$

Theorem 1.0.4. *On any smooth manifold M there exists a smooth vector field which is transverse to the zero section.*

Let M^n be a compact smooth manifold without boundary. The *zero section* $0(M)$ is the set of all zero vectors in TM i.e. $0(M) = \{(p, v) \in TM : v = 0 \in T_pM\}$.

Let V be a smooth vector field on M which is transverse to the zero section $0(M)$. Then $V^{-1}(0(M))$ is a submanifold of M dimension 0 i.e. it's a discrete set of isolated points in M. Moreover, it is closed as a preimage of a closed set. Hence it's compact and therefore finite. Thus, a smooth vector field transverse to $0(M)$ has finitely many zeros.

Definition 1.0.5. Let M^n be a compact smooth manifold without boundary. Let V be a smooth vector field on M such that $V \pitchfork 0(M)$. Define the Euler characteristic of M mod 2 to be the number of zeros of V mod 2. I.e. $\chi(M)$ mod $2 = 0$ if V has even number of zeros and $\chi(M)$ mod $2 = 1$ if V has odd number of zeros.

Theorem 1.0.6. *Let M^n be a compact smooth manifold without boundary. Then $\chi(M)$ mod 2 is well defined.*

Sketch of the Proof. Let V_0, V_1 be two different smooth vector fields on M which are transverse to the zero section. We need to show that $\#\{V_0 = 0\}$ and $\#\{V_1 = 0\}$ have the same parity.

Let $F\colon M \times [0,1] \to TM$ be the homotopy between V_0 and V_1 given by $F(p,t) = (1-t)V_1(p) + tV_2(p)$. Then F is a smooth map and it's transverse to $0(M)$ on its boundary $\partial(M \times [0,1]) = M \times \{0\} \cup M \times \{1\}$. By the Transversality Aproximation Theorem F is homotopic to a map $\tilde{F}\colon M \times [0,1] \to TM$ such that $\tilde{F} \pitchfork 0(M)$ and $\tilde{F}|_{\partial(M\times[0,1])} = F|_{\partial(M\times[0,1])}$, i.e. $\tilde{F}(p,0) = V_0(p)$ and $\tilde{F}(p,1) = V_1(p)$ for any $p \in M$.

Then the set $S = \tilde{F}^{-1}(0(M)) \subset M \times [0,1]$ is a compact submanifold with boundary of dimension 1 and $\partial S \subset \partial(M \times [0,1]) = M \times \{0\} \cup M \times \{1\}$. By construction, $\partial S \cap M \times \{0\} = V_0^{-1}(0)$ and $\partial S \cap M \times \{1\} = V_0^{-1}(1)$

Since S is compact and 1-dimensional it's diffeomorphic to a disjoint union of finitely many circles and closed intervals. Since a closed interval has two boundary point we have that $\#\partial S$ is **even**. On the other hand, by above, $\#\partial S = \#\{V_0 = 0\} + \#\{V_1 = 0\}$. Therefore, $\#\{V_0 = 0\}$ and $\#\{V_1 = 0\}$ have the same parity. $\square$

Example 1.0.7. Let $M = \mathbb{S}^1$. The rotation vector field $V(x,y) = (-y,x)$ is tangent to S^1 and has no zeros and hence $\chi(S^1)$ mod $2 = 0$.

Example 1.0.8. Let M, N be compact manifolds. Suppose $\chi(M)$ mod $2 = 0$. Then $\chi(M \times N)$ mod $2 = 0$.

Example 1.0.9. Let $M = \mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Let $V(x,y,z) = (-y,x,0)$.

Then V is a smooth tangent vector field on S^2 and it's transverse to the zero section. Since V has two zeros at $(0,0,\pm 1)$ we can conclude that $\chi(\mathbb{S}^2)$ mod $2 = 0$.

Example 1.0.10. Let $M = \mathbb{RP}^2 = \mathbb{S}^2/\pm 1$. Let $\pi\colon \mathbb{S}^2 \to \mathbb{RP}^2$ be the canonical projection.

Let $V(x,y,z) = (-y,x,0)$ be the vector field from the previous example. Let $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the antipodal map $A(x,y,z) = (-x,-y,-z)$. Since A is a linear map, it's equal to its differential at any point $(x,y,z) \in R^3$. Thus for any $p = (x,y,z) \in \mathbb{S}^2$ we have that $dA_p(V(p)) = -V(p) = (-x,-y,0) = V(-p)$. Since $\pi = \pi \circ A$, by the chain rule we have $d\pi = d\pi \circ dA$ and hence $d\pi_p(V(p)) = d\pi|_{-p}(V(-p))$ which means that V induces a well defined vector field $\tilde{V}$ on $\mathbb{RP}^2$. Since π is a local diffeomorphism, $\tilde{V}$ is smooth and transverse to the zero section. By construction, $\tilde{V}$ has exactly one zero as $[0:0:1] = [0:0:-1]$.

Therefore, $\chi(\mathbb{RP}^2)$ mod $2 = 1$.

2. Integral curves and integral flows

Definition 2.0.1. Let M^n be a compact smooth manifold without boundary. Let V be a smooth vector field on M. Let $\gamma\colon | \to M$ be a smooth curve in M where $I \subset \mathbb{R}$ is an interval. γ is called an integral curve of V if $\gamma'(t) = V(\gamma(t))$ for any $t \in I$.

(2.0.1)
If $\gamma(t)$ is an integral curve of V then $\gamma(t+c)$ is also an integral curve of V.

Let $M = U \subset \mathbb{R}^n$ be an open subset. Let $V(x) = (V_1(x), \dots, V_n(x))$ be a smooth vector field on M. Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be a curve in M. Then γ is an integral curve of V iff it satisfies the ODE

$$\begin{cases} x_1' = V_1(x) \\ \dots \\ x_n' = V_n(x) \end{cases}$$

The uniqueness and existence theorem for solutions of ODEs imples

Theorem 2.0.2. *Let V be a smooth vector field on M. Let $p \in M$ be any point. Then there exists $\varepsilon > 0$ such that there is a unique integral curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ of V such that $\gamma(0) = p$. Moreover, the same ϵ works for points q sufficiently close to p.*

Example 2.0.3. Let $M = \mathbb{R}^2 \backslash \{0\}$ and $V(x, y) = (1, 0)$. Let $p = (-1, 0)$. Then $\gamma(t) = (-1 + t, 0)$ is the unique integral curve starting at p. It's maximal existence interval is $(-\infty, 1)$.

Definition 2.0.4. A vector field V on M is called *complete* if all integral curves of V exist for all $t \in \mathbb{R}$.

Theorem 2.0.5. *let M be a compact manifold without boundary. Then any smooth vector field V on M is complete.*

Let V be a complete vector field. For any $t \in \mathbb{R}$ and $p \in M$ let $\varphi_t(p) = \gamma(t)$ where γ is the integral curve of V satisfying $\gamma(0) = p$.

Lemma 2.0.6. *(1) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for any $t, s \in \mathbb{R}$.*
(2) $\varphi_t\colon M \to M$ is a diffeomorphism for any $t \in \mathbb{R}$

Proof. Obviously, $\varphi_0 = \mathrm{id}_M$. Also, solutions of Initial Value Problems for ODEs depend smoothly on initial conditions the map φ_t is smooth for any t.

Let $p \in M, s \in \mathbb{R}$. Let $\gamma_1(t) = \varphi_t(\varphi_s(p))$ and $\gamma_2(t) = \varphi_{t+s}(p)$. Then $\gamma_1(0) = \varphi_s(p)$ and $\gamma_2(0) = \varphi_s(p)$. Also, $\gamma_1(t)$ is a integral curve of V by the definition of φ_t. $\gamma_2(t)$ is also an integral curve of V by the definition and (2.0.1). Therefore, $\gamma_1(t) = \gamma_2(t)$ by uniqueness of integral curves. This proves part (1). Part (2) is immediate from part (1) since

$$\varphi_{-t} \circ \varphi_t = \varphi_t \circ \varphi_{-t} = \varphi_0 = \mathrm{id}_M$$

□

The map $\varphi\colon \mathbb{R} \times M \to M$ is called the *flow* or *integral flow* of V and the maps $\varphi_t\colon M \to M$ are called the *flow maps* of V. Because solutions of Initial Value Problems for ODEs depend smoothly on initial conditions Lemma 2.0.6 immediately implies

Theorem 2.0.7. *Let V be a complete smooth vector field on M. Then its integral flow $\varphi\colon \mathbb{R} \times M \to M$ is a smooth action of $\mathbb{R}$ on M.*