1. Whitney Embedding

**Theorem 1.0.1.** Let $M^n$ be a smooth manifold. Then $M^n$ admits a smooth proper embedding into $\mathbb{R}^{2n+1}$.

**Proof.** We will only give a proof when $M$ is compact. It’s enough to show that there exists a smooth $1-1$ immersion $f: M \to \mathbb{R}^{2n+1}$. Such an $f$ will be automatically a proper embedding due to compactness of $M$.

**Lemma 1.0.2.** Let $M^n$ be a compact $n$-manifold. There exists a a smooth $1-1$ immersion $f: M \to \mathbb{R}^k$ for some large $k$ depending on $M$.

**Proof of Lemma 1.0.2.** For any point $p$ choose a coordinate ball $B'_p$ containing $p$ and let $x_p: B'_p \to V'_p$ be a local coordinate chart where $V'_p$ is an open ball in $\mathbb{R}^n$. Let $B_p \subset B'_p$ be a smaller coordinate ball containing $p$ and such that $x_p(B_p) = V_p$ is a concentric ball to $V'_p$ of smaller radius. Then $B_p$ is compact and is contained in $B'_p$. We have that $\cup_{p \in M} = M$ is an open cover of $M$. By compactness we can choose a finite subcover by coordinate balls $B_1, \ldots, B_m$ such that $x_i: B'_i \to V'_i, i = 1, \ldots, m$ are local coordinate charts. Let $\varphi_i: M \to \mathbb{R}$ be a smooth function such that $\varphi_i = 1$ on $B_i$, $\varphi_i = 0$ on $\bar{M}\setminus B'_i$ and $0 < \varphi_i < 1$ on $B'_i \setminus B_i$.

Let $f: M \to \mathbb{R}^{m(n+1)}$ be given by

$$f(p) = (\varphi_1(p)x_1(p), \ldots, \varphi_m(p)x_m(p), \varphi_1(p), \ldots, \varphi_m(p))$$

**Claim 1:** $f$ is $1-1$.

Let $p, q \in M$ be distinct points. If $p, q \in B_i$ for some $i$ then $\varphi_i(p) = \varphi_i(q) = 1$ but $x_i(p) \neq x_i(q)$ since $x_i$ is a diffeomorphism on $B'_i$ and in particular it’s $1$-$1$ on $B'_i$. Hence $\varphi_i(p)x_i(p) \neq \varphi_i(q)x_i(q)$ and therefore $f(p) \neq f(q)$.

If $p, q$ don’t belong to the same $B_i$ for any $i$ then there is an $i$ such that $p \in B_i$ but $q \notin B_i$. then $\varphi_i(p) = 1$ but $\varphi_i(q) < 1$ and we can again conclude that $f(p) \neq f(q)$.

**Claim 2:** $f$ is an immersion.

Let $p \in M, v \in T_p M, v \neq 0$. Then $p \in B_i$ for some $i$. $B_i$ is open in $M$ and $\varphi_i = 1$ on $B_i$. Therefore $d(\varphi_i \circ x_i)|_p(v) = d(x_i)|_p(v) \neq 0$ since $x_i$ is an immersion on $B_i$. Therefore $df_p(v) \neq 0$ and hence $F$ is an immersion. This finishes the proof of Claim 2 and hence of Lemma 1.0.2.

**Lemma 1.0.3.** Let $S \subset \mathbb{R}^N$ be a submanifold of dimension $n$ where $N > 2n+1$. For any nonzero $v \in \mathbb{R}^N$ let $\pi_v: \mathbb{R}^N \to v^{\perp} \cong \mathbb{R}^{N-1}$ be the orthogonal projection. Then there exists $v \neq 0$ such that $\pi_v|_S: S \to \mathbb{R}^{N-1}$ is a 1-1 immersion.

**Proof of Lemma 1.0.3.** Note that $\ker \pi_v = \mathbb{R}v$ for any $v \neq 0$.

Observe that if $\pi_v(p) = \pi_v(q)$ for some distinct $p, q \in S$ then $p - q = \lambda v$ for some nonzero $\lambda$ i.e. $[p - q] = [v] \in \mathbb{R}^{P^{N-1}}$. Also, since $\pi_v$ is linear, $d(\pi_v)|_p = \pi_v$ for any $p \in \mathbb{R}^n$. Therefore, if $(p, u) \in T_p S, u \neq 0$ and $d(\pi_u)|_p(u) = 0$ then $u = \lambda v$ and $[u] = [v] \in \mathbb{R}^{P^{N-1}}$. 


Let $h: S \times S \setminus \Delta S \to \mathbb{R}P^{N-1}$ be given by $h(p, q) = [p - q]$ and $g: TS \setminus 0(S) \to \mathbb{R}P^{N-1}$ be given by $g(p, u) = [u]$. Since $N - 1 > 2n = \dim S \times S = \dim TS$, the only way $[v] \in \mathbb{R}P^{N-1}$ can be a regular value of $h \ (g)$ is if $[v] \notin$ image of $h \ (g)$.

By Sard’s theorem the set of singular values of $h \ (g)$ has measure 0 and since the union of two sets of measure zero has measure zero we can find $[v] \in \mathbb{R}P^{N-1}$ which is a regular value for both $g$ and $h$. By above that means that $[v]$ is not in the image of either $h$ or $g$. Again, by above that means that

\[ \pi_v|S: S \to \mathbb{R}^{N-1} \text{ is a 1-1 immersion.} \]

To finish the proof of Whitney’s theorem we repeatedly apply Lemma 1.0.3 to $S = f(M)$ provided by Lemma 1.0.2.

2. Transversality

**Definition 2.0.1.** Let $S_1, S_2$ be submanifolds in a smooth manifold $M^n$. We say that $S_1$ is transverse to $S_2$ (denoted by $S_1 \pitchfork S_2$) if for any $p \in S_1 \cap S_2$ it holds that $T_p S_1 + T_p S_2 = T_p M$.

**Definition 2.0.2.** Let $S$ be a submanifold in a smooth manifold $N^n$. Let $f: M \to N$ be a smooth map. We say that $f$ is transverse to $S$ (denoted by $f \pitchfork S$) if for any $p \in f^{-1}(S)$ it holds that $T_{f(p)} S + df_p(T_p M) = T_p N$.

- Let $S_1, S_2$ be submanifolds in $M$. Then $S_1 \pitchfork S_2$ iff $i_1 \pitchfork S_2$ where $i_1: S_1 \to M$ is the canonical inclusion.
- Let $f: M \to N$ be smooth and let $S = c \in N$ be a point which we view as a zero-dimensional submanifold. Then $f \pitchfork S$ iff $c$ is a regular value of $f$.
- Let $S_1, S_2$ be submanifolds in $M$ such that $\dim S_1 + \dim S_2 < \dim M$. Then $S_1 \pitchfork S_2$ iff $S_1 \cap S_2 = \emptyset$.
- If $f: M \to N$ is a submersion then $f \pitchfork S$ for any submanifold $S$ in $N$.

**Example 2.0.3.**

- Let $M = \mathbb{R}^n$, $S_1 = \mathbb{R}^k \times 0$, $S_2 = 0 \times \mathbb{R}^l$. Then $S_1 \pitchfork S_2$ iff $k + l \geq n$.
- Let $M = \mathbb{R}^2$, $S_1 = \mathbb{R} \times 0$, $S_2 = \{(x, y): y = x^2\}$. Then $S_1$ is not transverse to $S_2$.
- Let $M = \mathbb{R}^n$, $S_1 = S^{n-1}$ - the unit sphere centered at 0 and let $S_2 = V$ be a vector subspace of $\mathbb{R}^n$. Then $S_1 \pitchfork V$.

**Theorem 2.0.4.** Let $f: M \to N$ be smooth and transverse to a submanifold $S$ in $N$. Then $P = f^{-1}(S)$ is a submanifold of $M$ if the same codimension as $S$, i.e. $\dim N - \dim S = \dim M - \dim P$.

Moreover, if $M$ has a boundary and $f|\partial M$ is still transverse to $S$ then $P = f^{-1}(S)$ is a submanifold with boundary in $M$. 

Definition 2.0.5. Two maps $f_0, f_1 : X \to Y$ are called homotopic ($f_0 \sim f_1$) if there exists a continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for any $x \in X$.

Theorem 2.0.6. Let $M, N$ be smooth manifolds and let $f_0, f_1 : M \to N$ be smooth maps. Then $f_0$ is continuously homotopic to $f_1$ iff $f_0$ is smoothly homotopic to $f_1$ i.e. iff there is a smooth map $F : M \times [0, 1] \to N$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for any $x \in M$.

Theorem 2.0.7 (Whitney approximation theorem). Let $M, N$ be smooth manifolds and let $f : M \to N$ be a continuous map. Then $f$ is homotopic to a smooth map $g : M \to N$. Moreover, if $f$ is smooth on a closed subset $A \subset M$ then $g$ can be chosen so that $g|_A = f|_A$.

Theorem 2.0.8 (Transversality Approximation theorem). Let $M, N$ be smooth manifolds and let $S \subset N$ be a submanifold. Let $f : M \to N$ be a smooth map. Then $f$ is homotopic to a smooth map $g : M \to N$ such that $g \pitchfork S$.

Moreover if $T \subset M$ is a submanifold and $f|_T$ is transverse to $S$ then $g$ can be chosen so that $g|_T = f|_T$. 