1. Whitney embedding

Theorem 1.0.1. *Let M^n be a smooth manifold. Then M^n admits a smooth proper embedding into $\mathbb{R}^{2n+1}$.*

Proof. We will only give a proof when M is compact. It's enough to show that there exists a smooth $1-1$ immersion $f\colon M \to \mathbb{R}^{2n+1}$. Such an f will be automatically a proper embedding due to compactness of M.

Lemma 1.0.2. *Let M^n be a compact n-manifold. There exists a a smooth $1-1$ immersion $f\colon M \to \mathbb{R}^k$ for some large k depending on M.*

Proof of Lemma 1.0.2. For any point p choose a coordinate ball B'_p containing p and let $x_p\colon B'_p \to V'_p$ be a local coordinate chart where V'_p is an open ball in $\mathbb{R}^n$. Let $B_p \subset B'_p$ be a smaller coordinate ball containing p and such that $x_p(B_p) = V_p$ is a concentric ball to V'_p of smaller radius. Then $\bar{B}_p$ is compact and is contained in B'_p. We have that $\cup_{p\in M} = M$ is an open cover of M. By compactness we can choose a finite subcover by coordinate balls $B_1, \ldots, B_m$ such that $x_i\colon B'_i \to V'_i, i = 1, \ldots m$ are local coordinate charts. Let $\varphi_i\colon M \to R$ be a smooth function such that $\varphi_i = 1$ on $\bar{B}_i$, $\varphi_i = 0$ on $M\backslash B'_i$ and $0 < \varphi_i < 1$ on $B'_i\backslash\bar{B}_i$.

Let $f\colon M \to \mathbb{R}^{m(n+1)}$ be given by

$$f(p) = (\varphi_1(p)x_1(p), \ldots, \varphi_m(p)x_m(p), \varphi_1(p), \ldots, \varphi_m(p))$$

Claim 1: f is $1-1$.

Let $p, q \in M$ be distinct points. If $p, q \in \bar{B}_i$ for some i then $\varphi_i(p) = \varphi_i(q) = 1$ but $x_i(p) \neq x_i(q)$ since x_i is a diffeomoprhism on B'_i and in particular it's 1-1 on B'_i. Hence $\varphi_i(p)x_i(p) \neq \varphi_i(q)x_i(q)$ and therefore $f(p) \neq f(q)$.

If p, q don't belong to the same $\bar{B}_i$ for any i then there is an i such that $p \in B_i$ but $q \notin \bar{B}_i$. then $\varphi_i(p) = 1$ but $\varphi_i(q) < 1$ and we can again conclude that $f(p) \neq f(q)$.

Claim 2: f is an immersion.

Let $p \in M, v \in T_pM, v \neq 0$. Then $p \in B_i$ for some i. B_i is open in M and $\varphi_i = 1$ on B_i. Therefore $d(\varphi_i \cdot x_i)|_p(v) = d(x_i)|_p(v) \neq 0$ since x_i is an immersion on B_i. Therefore $df_p(v) \neq 0$ and hence F is an immersion. This finishes the proof of Claim 2 and hence of Lemma 1.0.2. □

Lemma 1.0.3. *Let $S \subset \mathbb{R}^N$ be a submanifold of dimension n where $N > 2n+1$. For any nonzero $v \in \mathbb{R}^N$ let $\pi_v\colon \mathbb{R}^N \to v^\perp \cong \mathbb{R}^{N-1}$ be the orthogonal projection. Then there exists $v \neq 0$ such that $\pi_v|_S\colon S \to \mathbb{R}^{N-1}$ is a 1-1 immersion.*

Proof of Lemma 1.0.3. Note that $\ker \pi_v = \mathbb{R}v$ for any $v \neq 0$.

Observe that if $\pi_v(p) = \pi_v(q)$ for some distinct $p, q \in S$ then $p - q = \lambda v$ for some nonzero λ i.e. $[p-q] = [v] \in \mathbb{RP}^{N-1}$. Also, since π_v is linear, $d(\pi_v)_p = \pi_v$ for any $p \in \mathbb{R}^n$. Therefore, if $(p, u) \in T_pS, u \neq 0$ and $d(\pi_v)_p(u) = 0$ then $u = \lambda v$ and $[u] = [v] \in \mathbb{RP}^{N-1}$.

Let $h\colon S\times S\backslash\Delta S\to\mathbb{RP}^{N-1}$ be given by $h(p,q)=[p-q]$ and $g\colon TS\backslash 0(S)\to\mathbb{RP}^{N-1}$ be given by $g(p,u)=[u]$. Since $N-1>2n=\dim S\times S=\dim TS$, the only way $[v]\in\mathbb{RP}^{N-1}$ can be a regular value of h (g) is if $[v]\notin$ image of h (g).

By Sards theorem the set of singular values of h (g) has measure 0 and since the union of two sets of measure zero has measure zero we can find $[v]\in\mathbb{RP}^{N-1}$ which is a regular value for both g and h. By above that means that $[v]$ is not in the image of either h or g. Again, by above that means that

$\pi_v|_S\colon S\to\mathbb{R}^{N-1}$ is a 1-1 immersion. □

To finish the proof of Whitney's theorem we repeatedly apply Lemma 1.0.3 to $S=f(M)$ provided by Lemma 1.0.2. □

2. Transversality

Definition 2.0.1. Let S_1, S_2 be submanifolds in a smooth manifold M^n. We say that S_1 is transverse to S_2 (denoted by $S_1\pitchfork S_2$) if for any $p\in S_1\cap S_2$ it holds that $T_pS_1+T_pS_2=T_pM$.

Definition 2.0.2. Let S be a submanifold in a smooth manifold N^n. Let $f\colon M\to N$ be a smooth map. We say that f is transverse to S (denoted by $f\pitchfork S$) if for any $p\in f^{-1}S$ it holds that $T_{f(p)}S+df_p(T_pM)=T_pN$.

- Let S_1, S_2 be submanifolds in M. Then $S_1\pitchfork S_2$ iff $i_1\pitchfork S_2$ where $i_1\colon S_1\to M$ is the canonical inclusion.
- Let $f\colon M\to N$ be smooth and let $S=c\in N$ be a point which we view as a zero-dimensional submanifold. Then $f\pitchfork S$ iff c is a regular value of f.
- Let S_1, S_2 be submanifolds in M such that $\dim S_1+\dim S_2<\dim M$. Then $S_1\pitchfork S_2$ iff $S_1\cap S_2=\emptyset$.
- If $f\colon M\to N$ is a submersion then $f\pitchfork S$ for any submanifold S in N.

Example 2.0.3.

- Let $M=\mathbb{R}^n$, $S_1=\mathbb{R}^k\times 0$, $S_2=0\times\mathbb{R}^l$.Then $S_1\pitchfork S_2$ iff $k+l\geqslant n$.
- Let $M=\mathbb{R}^2$, $S_1=\mathbb{R}\times 0, S_2=\{(x,y):y=x^2\}$. Then S_1 is not transverse to S_2.
- Let $M=\mathbb{R}^n$, $S_1=\mathbb{S}^{n-1}$ - the unit sphere centered at 0 and let $S_2=V$ be a vector subspace of $\mathbb{R}^n$. Then $S_1\pitchfork V$.

Theorem 2.0.4. *Let $f\colon M\to N$ be smooth and transverse to a submanifold S in N. Then $P=f^{-1}(S)$ is a submanifold of M if the same codimension as S, i.e.* $\dim N-\dim S=\dim M-\dim P$.

Moreover, if M has a boundary and $f|_{\partial M}$ is still transverse to S then $P=f^{-1}(S)$ is a submanifold with boundary in M.

Definition 2.0.5. Two maps $f_0, f_1\colon X \to Y$ are called homotopic ($f_0 \sim f_1$) if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for any $x \in X$.

Theorem 2.0.6. *Let M, N be smooth manifolds and let $f_0, f_1\colon M \to N$ be smooth maps. Then f_0 is continuously homotopic to f_1 iff f_0 is smoothly homotopic to f_1 i.e. iff there is a smooth map $F\colon M \times [0,1] \to N$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for any $x \in M$.*

Theorem 2.0.7 (Whitney approximation theorem)**.** *Let M, N be smooth manifolds and let $f\colon M \to N$ be a continuous map. Then f is homotopic to a smooth map $g\colon M \to N$. Moreover, if f is smooth on a closed subset $A \subset M$ then g can be chosen so that $g|_A = f|_A$.*

Theorem 2.0.8 (Transversality Approximation theorem)**.** *Let M, N be smooth manifolds and let $S \subset N$ be a submanifold. Let $f\colon M \to N$ be a smooth map. Then f is homotopic to a smooth map $g\colon M \to N$ such that $g \pitchfork S$.*

Moreover if $T \subset M$ is a submanifold and $f|_T$ is transverse to S then g can be chosen so that $g|_T = f_T$.