1. Tensor bundles on manifolds

Let M^n be a smooth manifold (possibly with boundary). Let $p \in M$ and let $x = (x^1, \ldots, x^n)\colon U \to V$ be local coordinate chart where $U \subset M$ is open, $V \subset \mathbb{R}^n$ is open and $p \in U$. Then $e_i = \frac{\partial}{\partial x_i}|_p$ $i = 1, \ldots n$ is a basis of T_pM. let $e^1, \ldots e^n$ be the dual basis of T_p^*M. Observe that for any smooth $f\colon U \to \mathbb{R}$ we have that $df_p\colon T_p \to \mathbb{R}$ is a linear map and thus it's an element of T_p^*M. Observe that for $f = x^j$ we have that $df_p(e_i) = df_p(\frac{\partial}{\partial x_i}_p) = \frac{\partial x^j}{\partial x_i}|_p = \delta_{ij}$. This means that $dx^1|_p, \ldots dx^n|_p$ is the dual basis $e^1, \ldots e^n$ of T_p^*M.

For any $\alpha \in T_p^*M$ we have that $\alpha = \sum_i \alpha_i e^i$ where $\alpha_i = \alpha(e_i)$. Applying this to $\alpha = df_p$ gives

$$df_p = \sum_i \frac{\partial f}{\partial x_i}(p) dx^i|_p \tag{1.0.1}$$

From now on we will always use the notation $dx^i|_p$ instead of e^i. If $\omega \in \Lambda^k(T_p^*M)$ is a k-form on T_pM we can express it as

$$\omega = \sum_{I = i_1 < \ldots < i_k} \omega_I dx^I \text{ where } dx^I = dx^{i_1}|_p \wedge \ldots \wedge dx^{i_k}|_p$$

We will denote $e_I = (\frac{\partial}{\partial x_{i_1}}|_p, \ldots, \frac{\partial}{\partial x_{i_1}}|_p)$ by $\frac{\partial}{\partial x_I}|_p$

Define $\Lambda^k(T^*M)$ to be the union $\cup_{p \in M} \Lambda^k(T_p^*M)$. Let $\pi\colon \Lambda^k(T^*M) \to M$ be the canonical projection $(p, \omega) \mapsto p$. Similarly to TM we can endow $\Lambda^k(T^*M)$ with a structure of a smooth manifold as follows.

Every element $w \in \Lambda^k(T_p^*M)$ has a unique representation in the form $w = \sum_{I = (i_1 < \ldots < i_k)} w_I dx^I|_p$. This gives a canonical bijection

$\Psi_x\colon \pi^{-1}(U) \to V \times \mathbb{R}^{\binom{n}{k}}$ given by

$$(p, \omega) \mapsto (x(p), \{w(\frac{\partial}{\partial x_I}|_p)\})$$

with the inverse map given by

$$(u, \{w_I\}_I) \mapsto (x^{-1}(u), \sum_{I = (i_1 < \ldots < i_k)} w_I dx^I|_{x^{-1}(u)})$$

Now suppose $y = (y^1, \ldots, y^n)\colon U_1 \to V_1$ is another local coordinate chart defined on a neighbourhood of p. It similarly gives a map $\Psi_y\colon \pi^{-1}(U_1) \to V_1 \times \mathbb{R}^{\binom{n}{k}}$ given by $(p, \omega) \mapsto (y(p), \{w(\frac{\partial}{\partial y_J}|_p\}_J)$.

We compute that the transition map $\Psi_y \circ \Psi_x^{-1}$ is given by

$$\Psi_y \circ \Psi_x^{-1}((u, \{w_I\}_I)) = \Psi_y((x^{-1}(u), \sum_{I = (i_1 < \ldots < i_k)} w_I dx^I|_{x^{-1}(u)})) =$$

$$= (y(x^{-1}(u)), \{w_J\}_J)$$

where for $J = j_1, \dots, j_k$ we have $w_J = \sum_{I=(i_1<\ldots<i_k)} w_I dx^I|_{x^{-1}(u)}(\frac{\partial}{\partial y_J})|_{x^{-1}(u)} = \sum_{I=(i_1<\ldots<i_k)} w_I \det(\frac{\partial y_{i_s}}{\partial x_{i_t}})(u)$.

We see that this map is smooth. The inverse map is given by a simililar formula and hence it's smooth as well. This defines a smooth structure on $\Lambda^k(T^*M)$.

Corollary 1.0.1. *The canonical projection $\pi\colon \Lambda^k(T^*M) \to M$ is a submersion.*

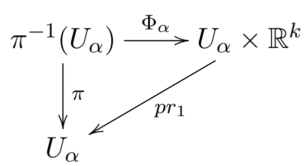
Remark 1.0.2. Note that by construction the coordinate maps Ψ_x commute with π and are linear isomorphisms on the fibers of π.

Similarly, one can define natural smooth structure on the space $\mathcal{T}^k(T^*M) = \cup_{p\in M}\mathcal{T}^k(T_p^*M)$ where $\mathcal{T}^k(T_p^*M)$ is the space of all (not necessarily alternating) k-tensors on T_pM.

2. Vector bundles

Let M^n be a smooth manifold (possibly with boundary). A smooth manifold (with boundary if M has boundary) E together with a smooth map $\pi\colon E \to M$ is called a smooth rank-k *vector bundle* over M if M admits an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that for each α there is a diffeomorphism

$\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that the following diagram commutes



(here $pr_1\colon U_\alpha \times \mathbb{R}^k \to U_\alpha$ is the canonical projection onto the first factor.)

and such that transition maps

$\Phi_\beta \circ \Phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ are linear isomorphisms on each $\mathbb{R}^k$ fiber. I.e. the transition maps have the form $(x, v) \mapsto (x, A(x)v)$ where $A(x)$ is a $k \times k$ invertible matrix depending smoothly on x.

Note that by construction, for any $p \in M$ the fiber $\pi^{-1}(p)$ carries a natural vector space structure.

Example 2.0.1. Let M^n be a smooth manifold (possibly with boundary). Then $TM \to M$, $\Lambda^k(T^*M) \to M$ and $\mathcal{T}^k(T^*M) \to M$ are smooth vector bundles.