1. Homotopy invariance of cohomology

Theorem 1.0.1 (Poincare lemma)**.** *Let M^n be a smooth manifold. Let $\pi\colon M\times\mathbb{R}\to M$ be the canonical projection. Let $t\in\mathbb{R}$ and let $s_t\colon M\to M\times\mathbb{R}$ be given by $s_t(p)=(p,t)$. Then the induced maps $s_t^*\colon H^*(M\times\mathbb{R})\to H^*(M)$ and $\pi^*\colon H^*(M)\to H^*(M\times\mathbb{R})$ are inverse to each other. In particular, both are isomorphisms.*

Corollary 1.0.2. *Under the assumptions of Theorem 1.0.1 for any $t_0,t_1\in\mathbb{R}$ it holds that*

$$s_{t_0}^*=s_{t_1}^*\colon H^*(M\times\mathbb{R})\to H^*(M).$$

Recall that continuous maps $f_0,f_1\colon X\to Y$ are called *homotopic* if there exists a continuous map $F\colon X\times[0,1]\to Y$ such that $F(x,0)=f_0(x), F(x,1)=f_1(x)$ for any $x\in X$. Homotopy is an equivalence relation on continuous maps from X to Y. We will denote it by $f_0\sim f_1$.

Theorem 1.0.3 (Homotopy invariance of cohomology)**.** *Let $f_0,f_1\colon M\to N$ be homotopic smooth maps between two manifolds (possibly with boundary). Then $f_0^*=f_1^*\colon H^*(N)\to H^*(M)$*

Proof. let $F\colon M\times[0,1]\to N$ be a homotopy from f_0 to f_1. let's extend it to a continuous map $\tilde F\colon M\times\mathbb{R}\to N$ in an obvious way by setting $\tilde F(x,t)=f_0(x)$ for $t\leqslant 0$ and $\tilde F(x,1)=f_1(x)$ for $t\geqslant 1$. Note that $\tilde F$ is smooth on $M\times\mathbb{R}\backslash[0,1]$. Therefore by a smooth approximation theorem we can find a smooth map $\bar F\colon M\times\mathbb{R}\to N$ such that $\bar F=\tilde F$ on $M\times\mathbb{R}\backslash[-1,2]$.

By construction we have that $\bar F\circ s_{-3}\equiv f_0$ and $\bar F\circ s_3\equiv f_1$. By Corollary 1.0.2 it holds that $s_3^*=s_{-3}^*\colon H^*(M\times\mathbb{R})\to H^*(M)$. Therefore $f_0^*=s_{-3}^*\circ\bar F^*=s_3^*\circ\bar F^*=f_1^*$. □

A map $f\colon X\to Y$ is called a *homotopy equivalence* if there exists $g\colon Y\to X$ such that $f\circ g\sim\mathrm{id}_Y$ and $g\circ f\sim\mathrm{id}_X$.

Example 1.0.4. A homeomorphism is a homotopy equivalence.

Definition 1.0.5. Let $A\subset X$. A map $r\colon X\to A$ is called a *retraction* if $r|_A=\mathrm{id}_A$. A retraction $r\colon X\to A$ is called a *deformation retraction* if $i\circ r\sim\mathrm{id}_X$ where $i\colon A\to X$ is the canonical inclusion. A deformation retraction $r\colon X\to A$ is called a *strong deformation retraction* if the homotopy F from $r\colon X\to X$ to id_X can be chosen to be identity on A i.e. it can be chosen to satisfy $F(x,t)=x$ for any $x\in A, t\in[0,1]$.

It's obvious that a deformation retraction is a homotopy equivalence.

Example 1.0.6. The map $f\colon\mathbb{R}^n\backslash\{0\}\to\mathbb{S}^{n-1}$ given by $f(x)=\frac{x}{|x|}$ is a strong deformation retraction.

Homotopy invariance of De Rham cohomology immediately implies

Corollary 1.0.7. *Let $f\colon M\to N$ be a homotopy equivalence.Then $f^*\colon H^*(N)\to H^*(M)$ is an isomorphism.*

Corollary 1.0.8. *Let M^n, N^m be closed orientable manifolds where $m \neq n$. Then M is not homeomorphic to N.*

Proof. Let $n < m$. Then $H^m(M) = 0$ for dimension reasons since any m-form on M is identically 0. But $H^m(N) \neq 0$ since $\int \colon H^m(N) \to \mathbb{R}$ is a non-zero homomorhism (we will later see that it's actually an isomorphism). □

Definition 1.0.9. A space X is called *contractible* if X is homotopy equivalent to a point.

Corollary 1.0.10. *Let M be a contractible manifold, possibly with boundary. Then*

$$H^*(M) \cong \begin{cases} \mathbb{R} \text{ if } * = 0 \\ 0 \text{ if } * > 0 \end{cases}$$

Definition 1.0.11. A subset $A \subset \mathbb{R}^n$ is called *star-shaped* if there exists $p \in A$ such that for any $x \in A$ the line segment $[px]$ is contained in A. In the is cases we will also say that A is star-shaped with respect to p.

Corollary 1.0.12. *Let $A \subset \mathbb{R}^n$ be star-shaped with respect to $p \in A$. Then p is a strong deformation retract of A. In particular, A is contractible and hence has the De Rahm cohomology of a point.*

Let $F \colon A \times [0,1] \to \mathbb{R}^n$ be given by $F(x,t) = (1-t)x + tp$. Since A is star-shaped with respect to p, we have that $F(x,t) \in A$ for any x, t and hence F is actually a map from $A \times [0,1]$ to A. By construction $F(x,0) = x$ and $F(x,1) = p$ for any $x \in A$.

Example 1.0.13. The following subsets of $\mathbb{R}^n$ are star-shaped with respect to 0. Hence the are contractible and have cohomology of a point.

- $A = \mathbb{R}^n$;
- $A = H^n$;
- $A = B(0,1)$.