1. Volume forms on Riemannian manifolds

Let (M^n, g) be a smooth oriented manifold of dimension n with a Riemannian metric g. Let $\omega = d\text{vol}_M$ be the volume form on M. Recall that it is defined as follows. For a point $p \in M$ let $e_1, \dots, e_n$ be a positive orthonormal basis of T_pM. Then $\omega_p = e^1 \wedge \dots \wedge e^n$. Note that this form is well-defined because if $\tilde{e}_1, \dots, \tilde{e}_n$ is another positive orthonormal basis of T_pM then $\omega_p = e^1 \wedge \dots \wedge e^n = \det A \cdot \tilde{e}_1 \wedge \dots \wedge \tilde{e}_n$ where A is the transition matrix from e to $\tilde{e}$. Since both bases are orthonormal $A \in O(n)$ and hence $\det A = \pm 1$. Since both bases are positive $\det A > 0$ and hence $\det A = 1$ and therefore $e^1 \wedge \dots \wedge e^n = \tilde{e}_1 \wedge \dots \wedge \tilde{e}_n$.

Let x be local coordinate chart near p positively oriented with respect to the orientation on M. oriented. let $g_{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i})(x)$ Then in coordinates x the volume form can be written as

$$d\text{vol}_M = \sqrt{\det(g_{ij})}dx^1 \wedge \dots \wedge dx^n \qquad (1.0.1)$$

This follows immediately from the following observation. Given vectors $v_1, \dots, v_n \in T_pM$ and a positive orthorormal basis $e_1, \dots e_n$ let A be the $n \times n$ matrix whose i-th column is given by the coordinates of v_i in the basis e. Then $|e^1 \wedge \dots \wedge e^n| = |\det A| = \sqrt{\det(A^tA)}$. Applying this to $v_1, \dots, v_n = \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ gives (1.0.1).

Note that since the volume form is by construction compatible with the orientation of M at every point we have that

$$\int_M d\text{vol}_M > 0.$$

Now, suppose $S \subset N$ is a submanifold of codimension 1. Let N be a unit normal vector field on S. Consider S with the orientation induced by normal field N and the orientation on M. Let g_S be the induced Riemannian metric on S.

Then the volume form on S can be given by the formula

$$d\text{vol}_S = i_N(d\text{vol}_M)$$

i.e. for any $p \in S$ and any $v_1, \dots, v_{n-1} \in T_pS$ we have

$$d\text{vol}_S(v_1, \dots, v_{n-1}) = (d\text{vol}_M)(N(p), v_1, \dots, v_{n-1})$$

2. Stokes's Theorem

Theorem 2.0.1 (Stokes' Theorem)**.** *Let M^n be an oriented n-dimensional manifold. let ω be a smooth$n-1$-form on M with compact support. then*

$$\int_{\partial M} \omega = \int_M d\omega$$

Example 2.0.2. Let $M = [0, 1]$ with the canonical orientation. Then $\partial M = +\{1\} - \{0\}$. Let $\omega = f \colon M \to \mathbb{R}$ be a smooth function. Then $\int_{\partial M} f = f(1) - f(0)$. On the other hand $d\omega = df = f'(x)dx$. Thus,

$\int_M d\omega = \int_0^1 f'(x)dx$. Hence, in this case Stokes's formula $\int_{\partial M}\omega = \int_M d\omega$ reduces to $f(1)-f(0) = \int_0^1 f'(x)dx$ which is the Fundamental Theorem of Calculus.

Example 2.0.3. Let $\omega = \frac{xdy-ydx}{2}$. Then $d\omega = dx\wedge dy$. Hence, by Stokes's formula for any compact domain D with smooth boundary in $\mathbb{R}^2$ we have

$$\int_{\partial D}\frac{xdy-ydx}{2} = \int_D dx\wedge dy = \text{Area }(D)$$

In particular, for $D=\{x^2+y^2\leqslant 1\}$ this gives $\int_{\partial D}\omega = \text{Area }(D)=\pi$.

Computing $\int_{\partial D}\omega$ using the parameterization $\varphi(t)=(\cos t,\sin t)$ we find that $\int_{\partial D}\omega = \int_{[0,2\pi)}\varphi^*\omega = \int_0^1\frac{dt}{2} = \pi$

Corollary 2.0.4. *Let ω be exact n-form on a compact oriented manifold M of dimension n. Then $\int_M\omega = 0$.*

Corollary 2.0.5. *Let ω be a closed $n-1$-form on a compact oriented manifold M of dimension n. Then $\int_{\partial M}\omega = 0$.*

Corollary 2.0.6. *Let M^n be an oriented manifold. Let ω be a closed k-form on M. Let $S\subset M$ be a compact oriented submanifold on M without a boundary. Suppose $\int_S\omega\neq 0$. Then*

(1) ω is not exact on M and $\omega|_S$ is not exact on S.
(2) S does not bound a compact oriented submanifold $N^{k+1}\subset M$.

Example 2.0.7. Let $M=\mathbb{R}^n\backslash\{0\}, S=\mathbb{S}^{n-1}=\{x_1^2+\dots x_n^2=1\}$, $\omega = \frac{\sum_i(-1)^{i-1}x_i dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots\wedge dx^n}{|x|^n}$. Then $\int_{S^{n-1}}\omega\neq 0$ and $d\omega = 0$. Therefore, the previous corollary applies. Hence ω is not exact on $\mathbb{R}^n\backslash\{0\}$ and $\mathbb{S}^{n-1}$ does not bound a compact oriented submanifold in $\mathbb{R}^n\backslash\{0\}$.

Let M^n be a compact oriented manifold without boundary. Observe that every n-form on M is closed for dimension reasons. Consider the map $I\colon\Omega^n(M)\to R$ given by $I(\omega)=\int_M\omega$. This map is obviously linear. By Corollary 2.0.4, exact forms line in the kernel of I. Therefore, I induces a liner map $I_*\colon H^n_{DR}(M)\to\mathbb{R}$. Note that for any orientation form ω on M we have that $I(\omega)>0$. therefore, I (and hence I_*) is onto.

Theorem 2.0.8. *Let M^n be a compact oriented manifold without boundary. Then $I_*\colon H^n_{DR}(M)\to\mathbb{R}$ is an isomorphism.*

We will prove this theorem later for general M. Let's show that it holds for $M=S^1$. We only need to check that $\ker I^*=0$. Let ω be 1-form on $\mathbb{S}^1$ such that $I(\omega)=\int_{\mathbb{S}^1}\omega=0$. We need to show that ω is exact.

Recall that $S^1=\mathbb{R}/\mathbb{Z}$ and we have a natural projection map $\pi\colon\mathbb{R}\to\mathbb{S}^1$ given by $\pi(t)=(\cos t,\sin t)$ which gives a diffeomorphism onto $S^1\backslash\{point\}$ when restricted to any interval $(a,a+2\pi)$. Therefore $\int_a^{a+2\pi}\pi^*\omega=\int_{\mathbb{S}^1}\omega=0$ for any $a\in\mathbb{R}$. We have $\pi^*\omega=u(t)dt$ for some smooth function $u(t)$ on $\mathbb{R}$. Define $f\colon\mathbb{R}\to\mathbb{R}$ by the formula, $f(x)=\int_0^x u(t)dt$. Then $df=\pi^*\omega$ and by above $f(a+2\pi)-f(a)=\int_a^{a+2\pi} u(t)dt=0$ for any real a. Thus, f is 2π-periodic. Therefore it induces a smooth function $\bar{f}\colon \mathbb{S}^1\to\mathbb{R}$ such that $\bar{f}\circ\pi=f$.

By construction $d\bar{f}=\omega$. $\square$.

3. Poincare lemma

Theorem 3.0.1. *Let M^n be a smooth manifold. Let $\pi\colon M\times\mathbb{R}\to M$ be the canonical projection and let $s\colon M\to M\times\mathbb{R}$ be the zero section given by $s(p)=(p,0)$. Then the induced maps $s^*\colon H^*(M\times\mathbb{R})\to H^*(M)$ and $\pi^*\colon H^*(M)\to H^*(M\times\mathbb{R})$ are inverse to each other. In particular, both are isomorphisms.*

Proof. Since $\pi\circ s=\mathrm{id}_M$ we obviously have that $s^*\circ\pi^*=\mathrm{id}_{H^*(M)}$. We need to show that $s^*\circ\pi^*=\mathrm{id}_{H^*(M\times\mathbb{R})}$ We will construct a *homotopy operator* $K\colon \Omega^*(M\times\mathbb{R})\to\Omega^{*-1}(M\times\mathbb{R})$ satisfying

$$\omega-(\pi^*\circ\mathfrak{s}^*)\omega=(-1)^{|\omega|-1}(dK-Kd)\omega \tag{3.0.1}$$

Let us first deal with the special case when $M=U$ is an open subset in $\mathbb{R}^n$. Let us define the operator K as follows. Let ω be a k-form on $U\times\mathbb{R}$. We can uniquely write it as

$$\omega=\sum_{I=(i_1<\ldots<i_k)} a_I(x,t)dx^I+\sum_{J=(j_1<\ldots<j_{k-1})} b_J(x,t)dx^J\wedge dt$$

Define $K(\omega)$ to be

$$K(\omega)=\sum_{J=(j_1<\ldots<j_{k-1})}(\int_0^t b_J(x,s)ds)dx^J$$

we claim that K satisfies (3.0.1). By linearity it's enough to check it for forms of the form $a(x,t)dx^I$ and $b(x,t)dx^J\wedge dt$.

Case 1. Let $\omega=a(x,t)dx^I$. Then $\pi^*\circ s^*(\omega)=a(x,0)dx^I$ and hence

$$\omega-\pi^*\circ s^*(\omega)=(a(x,t)-a(x,0))dx^I$$

By definition of K, $K(\omega)=0$. Also, $d\omega=da\wedge dx^I=\sum_i\frac{\partial a(x,t)}{\partial x^i}dx^i\wedge dx^I+\frac{\partial a(x,t)}{\partial t}dt\wedge dx^I=\sum_i\frac{\partial a(x,t)}{\partial x^i}dx^i\wedge dx^I+(-1)^k\frac{\partial a(x,t)}{\partial t}dx^I\wedge dt$. Therefore

$$Kd\omega=(-1)^k(\int_0^t\frac{\partial a(x,s)}{\partial s}ds)dx^I=(-1)^k(a(x,t)-a(x,0))dx^I$$

and

$$(-1)^{k-1}(dK-Kd)\omega=(-1)^{k-1}(-Kd)\omega=(a(x,t)-a(x,0))dx^I$$

which verifies (3.0.1).

Case 2. Now suppose $\omega=b(x,t)dx^J\wedge dt$. Then $s^*(\omega)=0$ and hence $\pi^*\circ s^*(\omega)=0$. Therefore

$$\omega-\pi^*\circ s^*(\omega)=\omega=b(x,t)dx^J\wedge dt$$

Next, $K\omega = (\int_0^t b(x,s)ds)dx^J$ and

$$dK\omega = b(x,t)dt \wedge dx^J + \sum_i \frac{\partial}{\partial x^i}\int_0^t (b(x,s)ds)dx^i \wedge dx^I =$$

$$= (-1)^{k-1}b(x,t)dx^J \wedge dt + \sum_i (\int_0^s \frac{\partial b(x,s)}{\partial x^i}ds)dx^i \wedge dx^I$$

On the other hand,

$$d\omega = \sum_i \frac{\partial b(x,t)}{\partial x^i}dx^i \wedge dx^I \wedge dt$$

and

$$Kd\omega = \sum_i (\int_0^t \frac{\partial b(x,s)}{\partial x^i}ds)dx^i \wedge dx^I$$

Therefore,

$$dK\omega - Kd\omega = (-1)^{k-1}b(x,t)dx^J \wedge dt$$

which again verifies (3.0.1).

Thus, we have proved that (3.0.1) holds for any k-form on $M \times \mathbb{R}$ when $M = U$ which is an open subset in $\mathbb{R}^n$. For a general M we can cover it by local coordinate charts $x_\alpha\colon U_\alpha \to V_\alpha$ and construct a subordinate partition of unity $\{\varphi_i\}_{i=1}^\infty$. Then any $\omega \in \Omega^*(M)$ can be written as $\omega = \sum_i \omega_i$ where $\omega_i = \varphi_i\omega$. then supp $\varphi_i\omega_i$ is contained in U_i and we already know how to define K for each ω_i because U_i is diffeomorphic to an open subset of $\mathbb{R}^n$. We can now define K by linearly extending it linearly:

$$K(\omega) :\overset{def}{=} \sum K(\omega_i)$$

It's immediate to check that K still satisfies (3.0.1).

Now, let ω be any closed form on M, i.e $d\omega = 0$. then (3.0.1) gives that

$$\omega - (\pi^* \circ \mathfrak{s}^*)\omega = (-1)^{|\omega|-1}(dK - Kd)\omega = (-1)^{|\omega|-1}(dK)\omega + 0$$

which means that $\omega - (\pi^* \circ \mathfrak{s}^*)\omega$ is exact and therefore $[\omega] = [(\pi^* \circ \mathfrak{s}^*)\omega] \in H^*(M \times \mathbb{R})$. □