1. Collar neighbourhood theorem

**Definition 1.0.1.** Let $M^n$ be a manifold with boundary. Let $p \in \partial M$. A vector $v \in T_pM$ is called inward if for some local chart $x: U \rightarrow V$ where $U$ is open, $V \subset H^n$ is open and $x(p) \in \partial H^n$ when we write $v$ as $v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\big|_p$ it holds that $v_n > 0$.

**Lemma 1.0.2.** If $v \in T_pM$ is inward with respect to a chart $x$ then it’s inward with respect to any other chart $y$.

*Proof.* This was proved in homework 5, problem (5). \qed

**Lemma 1.0.3.** Let $M^n$ be a manifold with boundary. Then there exists a smooth vector field $V$ on $M$ such that $V(p)$ is inward for any $p \in \partial M$.

*Proof.* Observe that if $v_1, \ldots, v_k \in T_pM$ are inward, $\lambda_1, \ldots, \lambda_k \geq 0$ and $\sum_{i} \lambda_i > 0$ then $v = \sum_{i} \lambda_i v_i$ is also inward. Now the result follows by an easy application of partition of unity. \qed

**Remark 1.0.4.** If $\partial M$ is compact it’s easy to construct a vector field $V$ as above such that $\text{supp}V$ is a compact neighbourhood on $\partial M$. In particular integral flow $\varphi_t$ of $V$ is defined for all $t$, i.e. $V$ is complete.

**Theorem 1.0.5** (Collar neighbourhood theorem). Let $M^n$ be a smooth manifold with boundary. Then there exists a neighbourhood $U \subset M$ of $\partial M$ diffeomorphic to $\partial M \times [0,1)$

*Proof.* We will only give a proof in case $\partial M$ is compact. Let $V$ be a complete vector field on $M$ which is inward along $\partial M$ provided by the lemma above. Let $\varphi: \mathbb{R} \times \partial M \rightarrow M$ be the flow of $V$ restricted to the boundary and let $p \in \partial M$. Since $\varphi_0(x) = x$ for any $x$ we have that $d\varphi_{(0,p)}(0,v) = v$ for any $v \in T_p \partial M$. Also, by definition of the flow we have that $d\varphi_{(0,p)}(\frac{\partial}{\partial t},0) = V(p)$. Since $V(p)$ is inward and $\dim M = \dim \mathbb{R} \times \partial M = n$ this implies that $d\varphi_{(0,p)}$ is an isomorphism. Therefore, by the Inverse Function Theorem, there is an open neighbourhood $U_p \subset \partial M$ containing $p$ and $\varepsilon_p > 0$ such that $\varphi|_{[0,\varepsilon_p) \times U_p}$ is a diffeomorphism onto its image which is an open neighbourhood of $p$ in $M$. Since $\partial M$ is compact we can choose a finite subcover $\{U_i\}_{i=1}^N$ from the open cover $\partial M = \bigcup_{p \in \partial M} U_p$. Then for $\varepsilon = \min_i \varepsilon_i$ we have that $\varphi|_{[0,\varepsilon) \times \partial M}$ is a local diffeomorphism from $[0,\varepsilon) \times \partial M$ onto its image. Using compactness of $\partial M$ and arguing by contradiction it’s easy to see that there exists $0 < \varepsilon_1 < \varepsilon$ such that $\varphi|_{[0,\varepsilon_1) \times \partial M}$ is 1-1. This means that $W = \varphi([0,\varepsilon_1) \times \partial M)$ is the desired collar neighbourhood. \qed

**Corollary 1.0.6.** Suppose $M_1^n, M_2^n$ are smooth manifolds with boundary and $f: N_1^{n-1} \rightarrow N_2^{n-1}$ is a diffeomorphism between some connected components of $\partial M_1$ and $\partial M_2$ respectively. Then the space obtained by gluing $M_1$ to $M_2$ along $f$ is an $n$-dimensional manifold (possibly with boundary).

2. Tensors on vector spaces

Let $V$ be a finite dimensional vector space over $\mathbb{R}$.
Definition 2.0.1. A tensor of type \((k,l)\) on \(V\) is a map
\[
T: \underbrace{V \times \ldots \times V}_k \times \underbrace{V^* \times \ldots \times V^*}_l \rightarrow \mathbb{R}
\]
which is linear in every variable.

Example 2.0.2.
- Let \(v \in V\) be a vector. Then \(v\) defines a tensor \(T_v\) of type \((0,1)\) with the map \(T_v: V^* \rightarrow \mathbb{R}\) given by \(T_v(f) = f(v)\). The map \(v \mapsto T_v\) gives a linear isomorphism from \(V\) onto \((V^*)^* = \text{space of all tensors of type } (0,1)\).
- Let \(\langle \cdot, \cdot \rangle\) be an inner product on \(V\). Then it is a tensor of type \((2,0)\).
- Let \(V = \mathbb{R}^n\) and let \(T: \underbrace{V \times \ldots \times V}_n \rightarrow \mathbb{R}\) be given by \(T(v_1, \ldots, v_n) = \det A\) where \(A\) is the \(n \times n\) matrix with columns \(v_1, \ldots, v_n\). Then \(T\) is a tensor of type \((n,0)\).

From now on we will only consider tensors of type \((k,0)\) which we’ll refer to as simply \(k\)-tensors. Let \(T^k(V)\) be the set of all \(k\) tensors on \(V\). It’s obvious that \(T^k(V)\) is a vector space and \(T^1(V) = V^*\). Also \(T^0(V) = \mathbb{R}\).

Definition 2.0.3. Let \(V, W\) be vector spaces and let \(L: V \rightarrow W\) be a linear map. Let \(T\) be a \(k\)-tensor on \(W\). Let \(L^*(T): \underbrace{V \times \ldots \times V}_k \rightarrow \mathbb{R}\) be defined by the formula
\[
L^*(T)(v_1, \ldots, v_k) := T(L(v_1), \ldots, L(v_k))
\]
Then it’s immediate that \(L^*(T)\) is a \(k\)-tensor on \(V\) which we’ll call the pullback of \(T\) by \(L\).

It’s easy to see that \(L^*: T^k(W) \rightarrow T^k(V)\) is linear.

Definition 2.0.4. Let \(T \in T^k(V)\), \(S \in T^l(V)\). We define their tensor product \(T \otimes S \in T^{k+l}(V)\) by the formula
\[
T \otimes S(v_1, \ldots, v_{k+l}) = T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+l})
\]
It’s obvious that \(T \otimes S\) is a tensor. The following properties of tensor product are obvious from the definition
- Tensor product is associative: \((T \otimes S) \otimes R = T \otimes (S \otimes R)\)
- Tensor product is linear in both variables: \((\lambda_1 T_1 + \lambda_2 T_2) \otimes R = \lambda_1 T_1 \otimes R + \lambda_2 T_2 \otimes R\) and the same holds for \(R\).
- Tensor product commutes with pullback, i.e. if \(L: V \rightarrow W\) is a linear map between vector spaces and \(T, S\) are tensors on \(W\) then
\[
L^*(T \otimes S) = L^*(T) \otimes L^*(S)
\]
Let us construct a basis of $T^k(V)$ and compute its dimension. Let $e_1, \ldots, e_n$ be a basis of $V$ and let $e^1, \ldots, e^n$ be the dual basis of $V^*$, i.e.

$$e^i(e_j) = \delta_{ij}$$

For any multi-index $I = (i_1, \ldots, i_k)$ with $1 \leq i_j \leq n$ define $\varphi^I$ as $\varphi^I = e^{i_1} \otimes \cdots \otimes e^{i_k}$. Also, we will denote the $k$-tuple $(e_{i_1}, \ldots, e_{i_k})$ by $e_I$.

It’s immediate from the definition that

$$(2.0.1) \quad \varphi^I(e_J) = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

For example $e^1 \otimes e^2(e_2, e_1) = e^1(e_2) \cdot e^2(e_1) = 0$.

**Lemma 2.0.5.** Let $T, S \in T^k(V)$ then $T = S$ iff $T(e_I) = S(e_I)$ for any multi-index $I = (i_1, \ldots, i_k)$.

**Proof.** This follows immediately from multi-linearity of $T$ and $S$. □

**Lemma 2.0.6.** The set $\{\varphi^I\}_{I=(i_1, \ldots, i_k)}$ is a basis of $T^k(V)$. In particular, $\dim T^k(V) = n^k$.

**Proof.** Let us first check linear independence of $\varphi^I$’s. Suppose $\sum_I \lambda_I \varphi^I = 0$. Let $J = (j_1, \ldots, j_k)$ be a multi-index of length $k$. Using (2.0.1) we obtain

$$0 = (\sum_I \lambda_I \varphi^I)(e_J) = \sum_I \lambda_I \varphi^I(e_J) = \sum_I \lambda_I \delta_{IJ} = \lambda_J$$

Since $J$ was arbitrary this proves linear independence of $\{\varphi^I\}_{I=(i_1, \ldots, i_k)}$.

Let us show that they span $T^k(V)$.

Let $T \in T^k(V)$. Let $S = \sum_I T(e_I) \varphi^I$. Then $S$ belongs to the span of $\{\varphi^I\}_I$. For any $J$ we have that $S(e_J) = \sum_I T(e_I) \varphi^I(e_J) = \sum_I T(e_I) \delta_{IJ} = T(e_J)$. Therefore, $T = S$ by Lemma 2.0.5 and hence $T$ belongs to the span of $\{\varphi^I\}_I$.

□

### 3. Alternating Tensors

**Definition 3.0.1.** Let $V$ be a finite dimensional vector space. A $k$-tensor $T$ on $V$ is called alternating if for any $v_1, \ldots, v_k$ and any $1 \leq i < j \leq k$ we have

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

**Example 3.0.2.**

- Any 1-tensor on $V$ is alternating.
- The determinant tensor defined in Example 2.0.2 is alternating.
- More generally, let $e_1, \ldots, e_n$ be a basis of $V$. Let $I = (i_1, \ldots, i_k)$ be a $k$-multi-index. Define a $k$ tensor $e^I$ as follows.
Let \( v_1, \ldots, v_k \in V \). Let \( A \) be the \( n \times k \) matrix whose \( i \)-th column is given by the coordinates of \( v_i \) with respect to the basis \( e_1, \ldots, e_n \).

Let \( A^I \) be the \( k \times k \) matrix made of rows \( i_1, \ldots, i_k \) of \( A \).

Define \( e^I \) by the formula

\[
e^I(v_1, \ldots, v_k) = \det A^I
\]

It’s immediate that \( e^I \) is alternating because the determinant of a matrix changes sign if two of its columns are switched. It’s also obvious that if \( I \) has some repeating indices then \( e^I = 0 \).

The following properties of alternating tensors are immediate from the definition

- Let \( \mathcal{A}^k(V) \) be the set of all alternating \( k \) tensors on \( V \). Then \( \mathcal{A}^k(V) \) is a vector subspace of \( \mathcal{T}^k(V) \), i.e. a linear combination of alternating tensors is alternating.
- If \( L : V \to W \) is linear and \( \omega \in \mathcal{A}^k(W) \) then \( L^*(\omega) \in \mathcal{A}^k(W) \)

**Remark 3.0.3.** In the notations of the book \( \mathcal{A}^k(V) = \Lambda^k(V^*) \).

Let \( \sigma \in S_k \) be a permutation and let \( T \in \mathcal{T}^k(A) \) be a \( k \)-tensor. Define \( \sigma T \) by

\[
\sigma T(v_1, \ldots, v_k) = T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})
\]

The following properties are immediate from the definition

- \( \sigma(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \sigma T_1 + \lambda_2 \sigma T_2 \)
- \( \sigma \tau T = \sigma(\tau T) \)

**Lemma 3.0.4.** Let \( T \in \mathcal{T}^k(V) \). Then TFAE

(i) \( T(v_1, \ldots, v_k) = 0 \) if \( v_i = v_j \) for some \( i \neq j \)
(ii) \( T \) is alternating;
(iii) \( T(v_1, \ldots, v_k) = 0 \) if \( v_1, \ldots, v_k \) are linearly dependent.
(iv) \( \sigma T = \sigma(\tau T) \) for any \( \sigma \in S_k \).

Let \( e_1, \ldots, e_n \) be a basis of \( V \). Let \( I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k) \) be two multi-indices with \( i_s \neq i_t, j_s \neq j_t \) for all \( s \neq t \).

It’s easy to see from the definition of \( e^I \) that \( e^I(e_J) = 0 \) if \( \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \) and \( e^I(e_J) = \text{sign } \sigma \) if \( \{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\} \) and \( \sigma \in S_k \) is the unique permutation satisfying \( I = \sigma(J) = (j_{\sigma(1)}, \ldots, j_{\sigma(k)}) \)

In particular, if \( I = (i_1 < i_2 < \ldots < i_k), J = (j_1 < j_2 < \ldots < j_k) \) then

\[
e^I(e_J) = \delta_{IJ}
\]

**Lemma 3.0.5.** Let \( \alpha, \beta \in \mathcal{A}^k(V) \). Then \( \alpha = \beta \) iff \( \alpha(e_I) = \beta(e_I) \) for any \( I = (i_1 < i_2 < \ldots < i_k) \).

**Lemma 3.0.6.** The set \( \{e^I\}_{I = (i_1 < i_2 < \ldots < i_k)} \) is a basis of \( \mathcal{A}^k(V) \). In particular \( \dim \mathcal{A}^k(V) = \binom{n}{k} \) for \( k \leq n \) and \( \dim \mathcal{A}^k(V) = 0 \) if \( k > n \).

**Proof.** The proof is the same as the proof of Lemma 2.0.6 but using Lemma 3.0.5 instead of Lemma 2.0.5 and (3.0.1) instead of (2.0.1). □
Example 3.0.7. Any element of $\mathcal{A}^k(V)$ can be written as a linear combination of the standard basis $\{\varphi^I\}$ of $\mathcal{T}^k(V)$. For example, if $n = \dim V = 4$ then $e^{13} = e^1 \otimes e^3 - e^3 \otimes e^1 = \varphi^{13} - \varphi^{31}$.

Lemma 3.0.8. Let $L : V \to V$ be a linear map and let $A = [L]$ be the matrix of $L$ with respect to the basis $e_1, \ldots, e_n$ of $V$. Then for any $w \in \mathcal{A}^n(V)$ we have that $L^*(w) = (\det A)w$.

Proof. By Lemma 3.0.6, $\mathcal{A}^n(V)$ is 1-dimensional with basis given by $e^{12\ldots n}$. Therefore, it’s enough to prove the lemma for $\omega = e^{12\ldots n}$.

We have $L^*(e^{12\ldots n})(e_1, \ldots, e_n) = e^{12\ldots n}(L(e_1), L(e_2), \ldots, L(e_n)) = \det A$ by definition of $e^{12\ldots n}$ and because columns of $A$ are given by $L(e_1), L(e_2), \ldots, L(e_n)$ written in the basis $(e_1, \ldots, e_n)$.

4. WEDGE PRODUCT

Lemma 4.0.1. Let $V$ be a finite-dimensional vector space. There exists a unique operation $\wedge : \mathcal{A}^k(V) \times \mathcal{A}^l(V) \to \mathcal{A}^{k+l}(V)$ satisfying the following conditions

i) $(\lambda_1 \omega_1 + \lambda_2 \omega_2) \wedge \eta = \lambda_1 \omega_1 \wedge \eta + \lambda_2 \omega_2 \wedge \eta$

ii) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$

iii) $\omega \wedge \eta = (-1)^{\vert \omega \vert \cdot \vert \eta \vert} \eta \wedge \omega$

iv) For any $\omega_1, \ldots, \omega_k \in V^*, v_1, \ldots, v_k \in V$ we have $\omega_1 \wedge \ldots \wedge \omega_k(v_1, \ldots, v_k) = \det(\omega_i(v_j))$

Sketch of proof. Let $e_1, \ldots, e_n$ be a basis of $V$ and let $e^1, \ldots, e^n$ be the dual basis of $V^*$. Let $\omega = \sum_i \omega_i e^i, \eta = \sum_j \omega_j e^j$. Define $\omega \wedge \eta$ by the formula

\[ \omega \wedge \eta := \sum_{I,J} \omega_I \eta_J e^{IJ} \]

It’s easy to see that so defined wedge product satisfies i)-iii). To see that it satisfies iv) check it for $\omega_1 = e^{i_1}, \ldots, \omega_k = e^{i_k}, v_1 = e_{j_1}, \ldots, v_k = e_{j_k}$. The general case follows by linearity.

This proves that a wedge product operation satisfying i)-iv) exists. It is also easy to see that because of iv) any such operation must satisfy the following: For any $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_l)$ it holds that

\[ e^I = e^{i_1} \wedge \ldots \wedge e^{i_k} \]

and

\[ e^I \wedge e^J = e^{IJ}. \]

Therefore, there is only one possible wedge product operation satisfying i)-iv). In particular, wedge product is well-defined and formula (4.0.1) produces the same answer irrespective of which basis of $V$ we use.

\[ \square \]