1. Collar neighbourhood theorem

Definition 1.0.1. Let M^n be a manifold with boundary. Let $p \in \partial M$. A vector $v \in T_pM$ is called *inward* if for some local chart $x\colon U \to V$ where U $subset M$ is open, $V \subset H^n$ is open and $x(p) \in \partial H^n$ when we write v as $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}|_p$ it holds that $v_n > 0$.

Lemma 1.0.2. *If $v \in T_pM$ is inward with respect to a chart x then it's inward with respect to any other chart y.*

Proof. This was proved in homework 5, problem (5). □

Lemma 1.0.3. *Let M^n be a manifold with boundary. Then there exists a smooth vector field V on M such that $V(p)$ is inward for any $p \in \partial M$.*

Proof. Observe that if $v_1, \ldots, v_k \in T_pM$ are inward, $\lambda_1, \ldots, \lambda_k \geqslant 0$ and $\sum_i \lambda_i > 0$ then $v = \sum_i \lambda_i v_i$ is also inward. Now the result follows by an easy application of partition of unity. □

Remark 1.0.4. *If ∂M is compact it's easy to construct a vector field V as above such that* $\operatorname{supp} V$ *is a compact neighbourhood on ∂M. In particular integral flow φ_t of V is defined for all t, i.e. V is complete.*

Theorem 1.0.5 (Collar neighbourhood theorem)**.** *Let M^n be a smooth manifold with boundary. Then there exists a neighbourhood $U \subset M$ of ∂M diffeomorphic to $\partial M \times [0, 1)$*

Proof. We will only give a proof in case ∂M is compact. Let V be a complete vector field on M which is inward along ∂M provided by the lemma above Let $\varphi\colon \mathbb{R} \times \partial M \to M$ be the flow of V restricted to the boundary and let $p \in \partial M$. Since $\varphi_0(x) = x$ for any x we have that $d\varphi_{(0,p)}(0, v) = v$ for any $v \in T_p\partial M$. Also, by definition of the flow we have that $d\varphi_{(0,p)}(\frac{\partial}{\partial t}, 0) = V(p)$. Since $V(p)$ is inward and $\dim M = \dim \mathbb{R} \times \partial M = n$ this implies that $d\varphi_{(0,p)}$ is an isomorphism. Therefore, by the Inverse Function Theorem, there is an open neighbourhood $U_p \subset \partial M$ containing p and $\varepsilon_p > 0$ such that $\varphi|_{[0,\varepsilon_p)\times U_p}$ is a diffeomorphism onto its image which is an open neighbourhood of p in M. Since ∂M is compact we can choose a finite subcover $\{U_i\}_{i=1}^N$ from the open cover $\partial M = \cup_{p\in\partial M} U_p$. Then for $\varepsilon = \min_i \varepsilon_i$ we have that $\varphi|_{[0,\varepsilon)\times\partial M}$ is a local diffeomorphism from $[0, \varepsilon) \times \partial M$ onto its image. Using compactness of ∂M and arguing by contradiction it's easy to see that there exists $0 < \varepsilon_1 < \varepsilon$ such that $\varphi|_{[0,\varepsilon_1)\times\partial M}$ is 1-1. This means that $W = \varphi([0, \varepsilon_1) \times \partial M)$ is the desired collar neighbourhood. □

Corollary 1.0.6. *Suppose M_1^n, M_2^n are smaooth manifolds with boundary and $f\colon N_1^{n-1} \to N_2^{n-1}$ is a diffeomorphism between some connected components of ∂M_1 and ∂M_2 respectively. Then the space obtained by gluing M_1 to M_2 along f is an n-dimensional manifold (possibly with boundary).*

2. Tensors on vector spaces

Let V be a finite dimensional vector space over $\mathbb{R}$.

Definition 2.0.1. A tensor of type (k, l) on V is a map

$$T\colon \underbrace{V \times \ldots \times V}_{k \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}} \to \mathbb{R}$$

which is linear in every variable.

Example 2.0.2.

- Let $v \in V$ be a vector. Then v defines a tensor T_v of type $(0, 1)$ with the map $T_v\colon V^* \to \mathbb{R}$ given by $T_v(f) = f(v)$. The map $v \mapsto T_v$ gives a linear isomorphism from V onto $(V^*)^*$ = space of all tensors of type $(0, 1)$.
- Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Then it is a tensor of type $(2, 0)$.
- Let $V = \mathbb{R}^n$ and let $T\colon \underbrace{V \times \ldots \times V}_{n \text{ times}} \to \mathbb{R}$ be given by $T(v_1, \ldots, v_n) = \det A$ where A is the $n \times n$ matrix with columns $v_1, \ldots, v_n$. Then T is a tensor of type $(n, 0)$.

From now on we will only consider tensors of type $(k, 0)$ which we'll refer to as simply *k-tensors.* Let $\mathcal{T}^k(V)$ be the set of all k tensors on V. It's obvious that $\mathcal{T}^k(V)$ is a vector space and $\mathcal{T}^1(V) = V^*$. Also $\mathcal{T}^0(V) = \mathbb{R}$.

Definition 2.0.3. Let V, W be vector spaces and let $L\colon V \to W$ be a linear map. Let T be a k-tensor on W. Let $L^*(T)\colon \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$ be defined by the formula

$$L^*(T)(v_1, \ldots, v_k) := T(L(v_1), \ldots, L(v_k))$$

Then it's immediate that $L^*(T)$ is a k-tensor on V which we'll call *the pullback of T by L.*

It's easy to see that $L^*\colon \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ is linear.

Definition 2.0.4. Let $T \in \mathcal{T}^k(V)$, $S \in \mathcal{T}^l(V)$. We define their *tensor product* $T \otimes S \in \mathcal{T}^{k+l}(V)$ by the formula

$$T \otimes S(v_1, \ldots, v_{k+l}) = T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+l})$$

It's obvious that $T \otimes S$ is a tensor. The following properties of tensor product are obvious from the definition

- Tensor product is associative: $(T \otimes S) \otimes R = T \otimes (S \otimes R)$
- tensor product is linear in both variables: $(\lambda_1 T_1 + \lambda_2 T_2) \otimes R = \lambda_1 T_1 \otimes R + \lambda_2 T_2 \otimes R$ and the same holds for R.
- tensor product commutes with pullback, i.e. if $L\colon V \to W$ is a linear map between vector spaces and T, S are tensors on W then

$$L^*(T \otimes S) = L^*(T) \otimes L^*(S)$$

Let us construct a basis of $\mathcal{T}^k(V)$ and compute its dimension. Let $e_1,\dots,e_n$ be a basis of V and let $e^1,\dots e^n$ be the dual basis of V^*, i.e.

$$e^i(e_j)=\delta_{ij}$$

For any multi-index $I=(i_1,\dots,i_k)$ with $1\leqslant i_j\leqslant n$ define φ^I as $\varphi^I=e^{i_1}\otimes\dots\otimes e^{i_k}$. Also, we will denote the k-tuple $(e_{i_1},\dots e_{i_k})$ by e_I.

It's immediate from the definition that

$$\varphi^I(e_J)=\delta_{IJ}=\begin{cases}1 \text{ if } I=J\\ 0 \text{ if } I\neq J\end{cases} \tag{2.0.1}$$

For example $e^1\otimes e^2(e_2,e_1)=e^1(e_2)\cdot e^2(e_1)=0$.

Lemma 2.0.5. *Let $T,S\in\mathcal{T}^k(V)$ then $T=S$ iff $T(e_I)=S(e_I)$ for any multi-index $I=(i_1,\dots i_k)$.*

Proof. This follows immediately from multi-linearity of T and S. □

Lemma 2.0.6. *The set $\{\varphi^I\}_{I=(i_1,\dots,i_k)}$ is a basis of $\mathcal{T}^k(V)$. In particular,* $\dim\mathcal{T}^k(V)=n^k$

Proof. Let us first check linear independence of φ^I's. Suppose $\sum_I\lambda_I\varphi^I=0$. Let $J=(j_1,\dots,j_k)$ be a multi-index of length k. Using (2.0.1) we obtain

$$0=(\sum_I\lambda_I\varphi^I)(e_J)=\sum_I\lambda_I\varphi^I(e_J)=\sum_I\lambda_I\delta_{IJ}=\lambda_J$$

Since J was arbitrary this proves linear independence of $\{\varphi^I\}_{I=(i_1,\dots,i_k)}$.

Let us show that they span $\mathcal{T}^k(V)$.

Let $T\in\mathcal{T}^k(V)$. Let $S=\sum_I T(e_I)\varphi^I$. Then S belongs to the span of $\{\varphi^I\}_I$. For any J we have that $S(e_J)=\sum_I T(e_I)\varphi^I(e_J)=\sum_I T(e_I)\delta_{IJ}=T(e_J)$. Therefore, $T=S$ by Lemma 2.0.5 and hence T belongs to the span of $\{\varphi^I\}_I$.

□

3. Alternating tensors

Definition 3.0.1. Let V be a finite dimensional vector space. A k-tensor T on V is called alternating if for any $v_1,\dots,v_k$ and any $1\leqslant i<j\leqslant k$ we have

$$T(v_1,\dots,v_i,\dots,v_j,\dots,v_k)=-T(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$$

Example 3.0.2.

- Any 1-tensor on V is alternating.
- The determinant tensor defined in Example 2.0.2 is alternating.
- More generally, let $e_1,\dots e_n$ be a basis of V. Let $I=(i_1,\dots,i_k)$ be a k-multi-index. Define a k tensor e^I as follows.

Let $v_1, \dots, v_k \in V$. Let A be the $n \times k$ matrix whose i-th column is given by the coordinates of v_i with respect to the basis $e_1, \dots e_n$. Let A^I be the $k \times k$ matrix made of rows $i_1, \dots, i_k$ of A.

Define e^I by the formula

$$e^I(v_1, \dots, v_k) = \det A^I$$

It's immediate that e^I is alternating because the determinant of a matrix changes sign if two of its columns are switched. It's also obvious that if I has some repeating indices then $e^I = 0$.

The following properties of alternating tensors are immediate from the definition

- Let $\mathcal{A}^k(V)$ be the set of all alternating k tensors on V. Then $\mathcal{A}^k(V)$ is a vector subspace of $\mathcal{T}^k(V)$, i.e. a linear combination of alternating tensors is alternating.
- If $L\colon V \to W$ is linear and $\omega \in \mathcal{A}^k(W)$ then $L^*(\omega) \in \mathcal{A}^k(W)$

Remark 3.0.3. In the notations of the book $\mathcal{A}^k(V) = \Lambda^k(V^*)$.

Let $\sigma \in S_k$ be a permutation and let $T \in \mathcal{T}^k(A)$ be a k-tensor. Define ${}^\sigma T$ by

$${}^\sigma T(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The following properties are immediate from the definition

- ${}^\sigma(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 {}^\sigma T_1 + \lambda_2 {}^\sigma T_2$
- ${}^{\sigma\tau}T = {}^\sigma({}^\tau T)$

Lemma 3.0.4. *Let $T \in \mathcal{T}^k(V)$. Then TFAE*

(i) $T(v_1, \dots, v_k) = 0$ *if* $v_i = v_j$ *for some* $i \neq j$
(ii) T *is alternating;*
(iii) $T(v_1, \dots, v_k) = 0$ *if* $v_1, \dots, v_k$ *are linearly dependent.*
(iv) ${}^\sigma T = \operatorname{sign}\sigma \cdot T$ *for any* $\sigma \in S_k$.

Let $e_1, \dots e_n$ be a basis of V. Let $I = (i_1, \dots i_k), J = (j_1, \dots, j_k)$ be two multi-indices with $i_s \neq i_t, j_s \neq j_t$ for all $s \neq t$.

It's easy to see from the definition of e^I that $e^I(e_J) = 0$ if $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$ and $e^I(e_J) = \operatorname{sign}\sigma$ if $\{i_1, \dots, i_k\} = \{j_1, \dots, j_k\}$ and $\sigma \in S_k$ is the unique permutation satisfying $I = \sigma(J) = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$

In particular, if $I = (i_1 < i_2 < \ldots < i_k), J = (j_1 < j_2 < \ldots < j_k)$ then

$$e^I(e_J) = \delta_{IJ} \tag{3.0.1}$$

Lemma 3.0.5. *Let $\alpha, \beta \in \mathcal{A}^k(V)$. Then $\alpha = \beta$ iff $\alpha(e_I) = \beta(e_I)$ for any $I = (i_1 < i_2 < \ldots < i_k)$.*

Lemma 3.0.6. *The set $\{e^I\}_{I=(i_1<i_2<\ldots<i_k)}$ is a basis of $\mathcal{A}^k(V)$. In particular $\dim \mathcal{A}^k(V) = \binom{n}{k}$ for $k \leqslant n$ and $\dim \mathcal{A}^k(V) = 0$ if $k > n$.*

Proof. The proof is the same as the proof of Lemma 2.0.6 but using Lemma 3.0.5 instead of Lemma 2.0.5 and (3.0.1) instead of (2.0.1). □

Example 3.0.7. Any element of $\mathcal{A}^k(V)$ can be written as a linear combination of the standard basis $\{\varphi^I\}$ of $\mathcal{T}^k(V)$. For example, if $n = \dim V = 4$ then $e^{13} = e^1 \otimes e^3 - e^3 \otimes e^1 = \varphi^{13} - \varphi^{31}$.

Lemma 3.0.8. *Let $L\colon V \to V$ be a linear map and let $A = [L]$ be the matrix of L with respect to the basis $e_1, \ldots, e_n$ of V. Then for any $w \in \mathcal{A}^n(V)$ we have that $L^*(w) = (\det A)w$.*

Proof. By Lemma 3.0.6, $\mathcal{A}^n(V)$ is 1-dimensional with basis given by $e^{12\ldots n}$. Therefore, it's enough to prove the lemma for $\omega = e^{12\ldots n}$.

We have $L^*(e^{12\ldots n}) = \lambda e^{12\ldots n}$ where $\lambda = L^*(e^{12\ldots n})(e_1, \ldots, e_n) = e^{12\ldots n}(L(e_1), L(e_2), \ldots, L(e_n)) = \det A$ by definition of $e^{12\ldots n}$ and because columns of A are given by $L(e_1), L(e_2), \ldots, L(e_n)$ written in the basis $(e_1, \ldots, e_n)$. □

4. Wedge product

Lemma 4.0.1. *Let V be a finite-dimensional vector space. There exists a unique operation $\wedge\colon \mathcal{A}^k(V) \times \mathcal{A}^l(V) \to \mathcal{A}^{k+l}(V)$ satisfying the following conditions*

i) $(\lambda_1\omega_1 + \lambda_2\omega_2) \wedge \eta = \lambda_1\omega_1 \wedge \eta + \lambda_2\omega_2 \wedge \eta$
ii) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$
iii) $\omega \wedge \eta = (-1)^{|\omega|\cdot|\eta|}\eta \wedge \omega$
iv) *For any* $\omega_1, \ldots, \omega_k \in V^*, v_1, \ldots, v_k \in V$ *we have* $\omega_1 \wedge \ldots \wedge \omega_k(v_1, \ldots, v_k) = \det(\omega_i(v_j))$

Sketch of proof. Let $e_1, \ldots, e_n$ be a basis of V and let $e^1, \ldots, e^n$ be the dual basis of $V^\star$ Let $\omega = \sum \omega_I e^I, \eta = \sum \omega_J e^J$. Define $\omega \wedge \eta$ by the formula

$$\omega \wedge \eta := \sum_{I,J} \omega_I \eta_J e^{IJ} \tag{4.0.1}$$

It's easy to see that so defined wedge product satisfies i)-iii). To see that it satisfies iv) check it for $\omega_1 = e^{i_1}, \ldots \omega_k = e^{i_k}, v_1 = e_{j_1}, \ldots v_k = e_{j_k}$. The general case follows by linearity.

This proves that a wedge product operation satisfying i)-iv) exists. It is also easy to see that because of iv) any such operation must satisfy the following: For any $I = (i_1, \ldots i_k), J = (j_1, \ldots, j_l)$ it holds that

$$e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$$

and

$$e^I \wedge e^J = e^{IJ}.$$

Therefore, there is only one possible wedge product operation satisfying i)-iv). In particular, wedge product is well-defined and formula (4.0.1) produces the same answer irrespective of which basis of V we use.

□