1. Differential forms on smooth manifolds

Definition 1.0.1. Let M^n be a smooth manifold (possibly with boundary). Let $\pi\colon E \to M$ be a smooth vector bundle over M. A *section* of E is a map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$. A section s is called smooth if it's smooth as a map $s\colon M \to E$.

Example 1.0.2. A smooth vector field on M is a smooth section of the tangent bundle $TM \to M$.

Definition 1.0.3. A differential k-form ω on M is a section of the bundle $\pi\colon \Lambda^k(T^*M) \to M$ of alternating k-tensors on M. I.e. ω is a map $\omega\colon M \to \Lambda^k(T^*M)$ such that $\omega(p) \in \Lambda^k(T^*_pM)$ for any $p \in M$.

Let U be an open subset in M^n and let $x = (x^1, \ldots, x^n)\colon U \to V$ be a local coordinate chart on M where V is an open subset in $\mathbb{R}^n$ (or H^n.

Then for any $p \in U$ the tangent space T_pM has a basis $e_1 = \frac{\partial}{\partial x_1}|_p, \ldots, e_n = \frac{\partial}{\partial x_n}|_p$. Therefore T^*_pM has a dual basis $e^1, \ldots, e^n$ were $e^i(e_j) = \delta_{ij}$.

Let $x^i\colon U \to \mathbb{R}$ be the i-th coordinate map. Consider $(dx^i)_p\colon T_pM \to \mathbb{R}$. Then $(dx^i)_p(\frac{\partial}{\partial x_j}|_p) = \frac{\partial x^i}{\partial x_j}|_p = \delta_{ij}$. In other words,

$$(dx^i)_p = e^i \qquad i = 1, \ldots, n$$

From now on we will use the notations $(dx_i)_p$ for the elements of the dual basis instead of e^i. However, these are the same objects and this is simply a notation change.

Similarly, instead of writing $e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$ we will write $dx^I|_p = dx^{i_1}|_p \wedge \ldots \wedge dx^{i_k}|_p$. Thus, every k-form ω on U can be uniquely written as

$$\omega = \sum_{I=(i_1<\ldots<i_k)} \omega_I(x) dx^I$$

. This gives a canonical bijection

Lemma 1.0.4. *Let $\omega = \sum_{I=(i_1<\ldots<i_k)} \omega_I(x) dx^I$ be a k-form on U. Then TFAE*

(1) ω is smooth as a map $U \to TU$
(2) ω_I is a smooth function on U for every $I = (i_1 < \ldots < i_k)$.
(3) For any smooth vector fields $V_1, \ldots, V_k$ on U it holds that $\omega(V_1(x), \ldots, V_k(x))$ is a smooth function on U.

We denote the set of all smooth k-form on M by $\Omega^k(M)$. We'll denote by $\Omega^*(M)$ the collection of all forms of all degrees i.e. $\cup_k \Omega^k(M)$.

Note that $\Omega^0(M) = C^\infty(M)$. All pointwise operations on alternating tensors such as addition, multiplication by a number and wedge product make sense for forms Moreover, if $\omega \in \Omega^k(M)$ and $f\colon M \to \mathbb{R}$ is smooth then $f \cdot \omega$ is also a smooth form.

Pullbacks make sense for forms as well.

Given a smooth map $f\colon M \to N$ and $\omega \in \Omega^k(N)$ we define $f^*(\omega)$ by $f^*(\omega)(p) = df_p^*(\omega(f(p)))$. I.e. for $v_1, \dots v_k \in T_pM$ we have $f^*\omega(p)(v_1, \dots, v_k) = \omega(f(p))(df_p(v_1), \dots, df_p(v_k))$. By computing $f^*(\omega)$ in local coordinates it follows from Lemma 1.0.4 that $f^*(\omega)$ is smooth.

Proposition 1.0.5.

a) If $\omega_1, \omega_2 \in \Omega^k(M)$, $f_1, f_2\colon M \to \mathbb{R}$ are smooth then $f_1\omega_1 + f_2\omega_2 \in \Omega^k(M)$ is also a smooth k-form.
b) If $\omega, \eta \in \Omega^*(M)$ then $\omega \wedge \eta \in \Omega^*(M)$
c) If $F\colon M \to N$ is smooth then $F^*\colon \Omega^*(N) \to \Omega^*(N)$ is linear. Moreover $F^*(g) = g \circ F$ for any $g \in \Omega^0(N)$.
d) If $F\colon M \to N$ and $G\colon N \to P$ are smooth and then $(G \circ F)^* = F^* \circ G^*$
e) If $f\colon M \to \mathbb{R}$ is smooth then $f^*(dt) = df$ - differential of f which in local coordinates x on M can be written as $\sum_i \frac{\partial f}{\partial x_i} dx^i$
f) If $F\colon M \to N$ is smooth and $\omega, \eta \in \Omega^*(N)$ then $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$
g) If $F = (F_1, \dots, F_m)\colon M \to \mathbb{R}^m$ is smooth then $F^*(\sum_{I=(i_1<\dots<i<k)} w_I(y)dy^I) = \sum_I (\omega_I \circ F) dF_{i_1} \wedge \dots \wedge dF_{i_k}$

Proof. a),b),c), f) are straightforward. d) follows from the definition of pullback and the chain rule $d(g \circ f) = dg \circ df$. e) is immediate from the definition: for $p \in M, v \in T_p(M)$ we have $f^*(dt)(v) = dt(df_p(v)) = df_p(v)$.

To get the coordinate expression for df recall that for any 1-form ω we have $\omega = \sum_i \omega(\frac{\partial}{\partial x_i}) dx^i$. In case of $\omega = df$ this gives $df = \sum_i df(\frac{\partial}{\partial x_i}) dx^i = \sum_i \frac{\partial f}{\partial x_i} dx^i$

g) follows from e), f):

$$F^*(\sum_{I=(i_1<\dots<i<k)} w_I(y)dy^I) = \sum_I F^*(w_I(y)dy^{i_1} \wedge \dots \wedge dy^{i_k}) = \sum_I (\omega_I \circ F) F^*(dy^{i_1}) \wedge \dots \wedge F^*(dy^{i_k}) =$$

$$\sum_I (\omega_I \circ F) dF^{i_1} \wedge \dots \wedge dF^{i_k}$$

□

Formula g) from the previous Proposition has a particularly simple form for top dimensional forms:

Lemma 1.0.6. *Let $F = (F_1, \dots, F_n)\colon U \to V$ be smooth where $U, V \subset \mathbb{R}^n$ are open. Let $\omega = u(y)dy^1 \wedge \dots dy^n$ be an n-form on V. Then*

$$F^*(\omega) = (u(F(x))(\det(\frac{\partial f_i}{\partial x_j}))dx^1 \wedge \dots \wedge dx^n$$

2. Exterior derivative

Proposition 2.0.1. *Let M^n be a smooth manifold (possibly with boundary). There exists a unique operation, called exterior derivative, $d\colon \Omega^*(M) \to \Omega^{*+1}(M)$ satisfying the following conditions*

a) Let $f\colon V \to \mathbb{R}$ be smooth. Then $df = df$, the differential of f.
b) $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is linear
c) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|\cdot|\eta|}\omega \wedge d\eta$
d) $d \circ d = 0$

Proof. $U \subset M$ be open and let $x = (x^1, \dots, x^n)\colon U \to V$ be a local coordinate chart. Let $\omega|_U = \sum_I \omega_I(x) dx^I$. Define $d\omega|_U$ by the formula

$$d\omega|_U = \sum_I d\omega_I(x) \wedge dx^I \tag{2.0.1}$$

We claim that so defined d satisfies a)-d) of the Proposition. a)-c) are straightforward. Let us verify d). For a 0-form $\omega = f$ we have $df = \sum_i \frac{\partial f}{\partial x_i} dx^i$. Then $d(df) = \sum_j \sum_i \frac{\partial^2 f}{\partial x_j \partial x_i} dx^j \wedge dx^i = \sum_{j<i} \frac{\partial^2 f}{\partial x_j \partial x_i} dx^j \wedge dx^i + \frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dx^j = \sum_{j<i} [\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}] dx^j \wedge dx^i = 0$.

For a general $\omega = \sum_I d\omega_I(x) \wedge dx^I$ we have $d\omega = \sum_I d\omega_I(x) \wedge dx^I$. By c) this implies $d(d\omega)) = \sum_I d(d\omega_I(x) \wedge dx^I) = \sum_I d(d\omega_i) \wedge dx^I + (-1)\omega_I \wedge d(dx^I) = 0 + 0 = 0$.

It's easy to see that conditions a)-d) implies that d must satisfy (2.0.1) in coordinates which proves uniqueness of d. Uniqueness of d also implies that we can use (2.0.1) to define d on global forms on M. □

Lemma 2.0.2. *Let $F\colon M \to N$ be a smooth map. Then $d \circ F = F \circ d$. I.e. for any $\omega \in \Omega^*(N)$ it holds*

$$F^*(d\omega) = d(F^*(\omega))$$

Proof. By Proposition 1.0.5 we know that if $\omega = dg$ where $g\colon N \to \mathbb{R}$ is smooth then $F^*(dg) = d(g \circ F))$.

Let y be some local coordinate son N

By linearity it's enough to prove the lemma for $\omega = u(y)dy^I$.

We have $d\omega = du \wedge dy^I$. Hence $F^*(d\omega) = F^*(du \wedge dy^I) = F^*(du) \wedge F^*(dy^I) = d(u \circ F) \wedge dF^{i_1} \wedge \dots \wedge dF^{i_k}$.

One the other hand, $F^*(\omega) = (u \circ F) dF^{i_1} \wedge \dots \wedge dF^{i_k}$. Then by repeatedly applying Proposition 2.0.1c0 and using that $d(dF_i) = 0$ we get that

$$dF^*(\omega) = d(u \circ F) \wedge dF^{i_1} \wedge \dots \wedge dF^{i_k}$$

□

3. De Rham cohomology

Definition 3.0.1. A form $\omega \in \Omega^*(M)$ is called *closed* if $d\omega = 0$.

A form $\omega \in \Omega^*(M)$ is called *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{*-1}(M)$.

Since $d \circ d = 0$ it's obvious that every exact form is closed. It's natural to ask to what extent the converse holds. Let $B^k(M)$ be the set of all exact k-forms and let $Z^k(M)$ be the set of all closed k forms. It's obvious that $B^k(M), Z^k(M)$ are vector spaces and by above $B^k(M) \subset Z^k(M)$.

Definition 3.0.2. Let M^n be a smooth manifold, possibly with boundary. The k-th de Rham cohomology group of M is defined to be the quotient group

$$H^k_{DR}(M) := Z^k(M)/B^k(M)$$

Since $B^k(M)$ is a vector subspace of $Z^k(M)$ the quotient $H^k_{DR}(M)$ is a vector space and not just a group.

By the definition that $H^k_{DR}(M) = 0$ iff every closed k-form is exact.

Example 3.0.3. Let $M = V$ be an open subset of $\mathbb{R}^2$. Then a 1-form ω on V has the form $P(x,y)dx + Q(x,y)dy$. By definition, w is exact iff $\omega = df$ for some smooth $f\colon V \to \mathbb{R}$, i.e. if $P(x,y)dx + Q(x,y)dy = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dxy$, or $P(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $Q(x,y) = \frac{\partial f}{\partial y}(x,y)$.

On the other hand ω is closed iff $0 = d\omega = d(P(x,y)dx + Q(x,y)dy) = (-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y))dx \wedge dy$ or $-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y) = 0$.

Thus, every closed 1-form on V is exact iff for any smooth $P, Q\colon V \to \mathbb{R}$ satisfying $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ there exists a smooth $f\colon V \to \mathbb{R}$ such that $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}(x,y)$.

Exercise 3.0.4. *Prove that* $H^1_{DR}(\mathbb{R}^2) = 0$

Let $f\colon M \to N$ be a smooth map between manifolds. Since F^* commutes with d, F^* sends closed forms to closed forms and exact forms to exact forms. Therefore it induces a homomorphism $F^*\colon H^k_{DR}(N) \to H^k_{DR}(M)$ for any k.

Since $(G \circ F)* = F^* \circ G^*$ and $Id^*_M = Id$ it follow that if $F\colon M \to N$ is a diffeomorphism then $F^*\colon H^k_{DR}(N) \to H^k_{DR}(M)$ is an isomorphism. We will see later that for $V = \mathbb{R}^2 \backslash \{0\}$ the form $\omega = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ is closed but not exact. This will imply that $H^1_{DR}(\mathbb{R}^2 \backslash \{0\}) \neq 0$. Since $H^1(\mathbb{R}^2) = 0$ by the exercise above, this will show that $\mathbb{R}^2$ is not diffeomorphic to $\mathbb{R}^2 \backslash \{0\}$.

4. Orientation

4.1. **Orientation on a vector space.**

Definition 4.1.1. Let V be a finite dimensional vector space. Let $e = (e_1, \dots, e_n)$ and $e' = (e'_1, \dots, e'_n)$ be two bases of V. We say that $e \sim e'$ if the transition matrix A from e to e' has $\det A > 0$. It's easy to see that $\sim$ satisfies the following properties

- if $e \sim e'$ then $e' \sim e$;
- if $e \sim e'$ and $e' \sim e''$ then $e \sim e''$.

This means that $\sim$ is an equivalence relation on the set of all bases of V. We will call equivalence classes mod $\sim$ *orientations* on V. We will say that two bases e, e' *have the same orientation* if they belong to the same equivalence class, i.e. the transition matrix from e to e' has positive determinant.

Lemma 4.1.2. *Let V be a finite dimensional vector space. Then there are precisely two possible ordinations on V.*

Proof. Let $e = (e_1, \dots, e_n)$ be a basis of V and let $e' = (-e_1, e_2 \dots, e_n)$. Since the transition matrix A from e to e' has determinant -1 they define two different orientations on V. We claim that any other basis of V is equivalent to either e or e': Let e'' be a basis of V. Let B be the transition matrix from e' to e''. Then the transition matrix from e to e'' is BA and $\det(BA) = \det B \cdot \det A = -\det B$. This means that $\det B$ and $\det(BA)$ have opposite signs, and thus one of them is positive and the other is negative. Therefore $e'' \sim e$ or $e'' \sim e'$. $\square$

We'll call the two distinct orientations on V *opposite or negative* to each other. If ϵ is an orientation and $e = (e_1, \dots, e_n)$ is a basis we put $\epsilon(e) = +1$ if e is positively oriented with respect to ϵ and we put $\epsilon(e) = -1$ if e is negatively oriented with respect to ϵ.

$\mathbb{R}^n$ has a canonical orientation defined by the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$.

Orientations on V correspond to orientations on $\mathcal{A}^n(V) \cong \mathbb{R}$ as follows.

Let $w \in \mathcal{A}^n(V)$ be a nonzero alternating n-tensor. It defines an orientation ϵ_w as follows:

Given a basis $e = (e_1, \dots, e_n)$ we'll say that e is positively oriented iff $w(e_1, \dots, e_n) > 0$. It's easy to see that this defines an orientation on V. It's also obvious that if $w' = \lambda w$ with $\lambda \neq 0$ then w and w' define the same orientation iff $\lambda > 0$.

4.2. **Orientation on manifolds.** Let M^n be a smooth n-dimensional manifold (possibly with boundary)

Definition 4.2.1. An orientation ϵ on M^n is a choice of orientation $\epsilon(p)$ on T_pM for all $p \in M$.

An orientation ϵ is called *continuous* if for any $p \in M$ there exists an open set $U \subset M$ containing p and a collection of continuous vector fields $X_1, \dots X_n$ on U such that $X_1(q), \dots X_n(q)$ is a basis of T_qM for any $q \in U$ and $\epsilon(X_1(q), \dots X_n(q)) = +1$ for any $q \in U$.

A manifold M is called *orientable* if it admits a continuous orientation.

Exercise 4.2.2. *Prove that an orientation ϵ is continuous if and only if it's smooth, i.e. for any $p \in M$ there exists an open set $U \subset M$ containing p and a collection of* **smooth** *vector fields $X_1, \dots X_n$ on U such that $X_1(q), \dots X_n(q)$ is a basis of T_qM for any $q \in U$ and $\epsilon(X_1(q), \dots X_n(q)) = +1$ for any $q \in U$.*

From now on we will only consider continuous orientations. The relation between orientations and nonzero alternating n-vectors on a fixed vector space naturally carries over to manifolds as follows.

Suppose ω is a smooth n-form on M^n such that $\omega(p) \neq 0$ for any $p \in M$. Then ω defines an orientation ϵ_ω on M as follows. Given $p \in M$ and a basis $v_1, \dots v_n$ of T_pM we say that it's positively oriented iff $\omega(p)(v_1, \dots v_n) > 0$.

Lemma 4.2.3. *ϵ_ω is continuous.*

Proof. Let $p \in M$ be any point. Let U be a coordinate ball containing p, so U is diffeomorphic to an open ball $B(0,1)$ in $\mathbb{R}^n$ under some local coordinate map $x\colon U \to \mathbb{R}^n$. Let $X_i(q) = \frac{\partial}{\partial x_i}(q)$. Then $f(q) = \omega(X_1(q), \dots, X_n(q))$ is smooth on U. Since $f(q) \neq 0$ for any q, by the Intermediate Value Theorem we must have that $f(q) > 0$ for all $q \in U$ or $f(q) < 0$ for all $q \in U$. In the first case this gives the required collection of continuous vector fields on U. In the second case the same works after changing X_1 to $-X_1$. $\square$

Next we will show that the converse also holds, i.e. every continuous orientation is equal to ϵ_ω for some nowhere zero $\omega \in \Omega^n(M^n)$.

Lemma 4.2.4. *Let ϵ be a continuous orientation on a smooth manifold M^n. Then there exists a smooth form $\omega \in \Omega^n(M)$ such that $\omega(p) \neq 0$ for any $p \in M$ and $\epsilon = \epsilon_\omega$.*

Proof. Let ϵ be a continuous orientation on M.

We will use the following terminology. Let ω be a smooth n-form on an open subset $U \subset M$. We will say that ω is *positive* on U if $\omega(p)(v_1, \dots, v_n) > 0$ for any $p \in U$ and any positive basis $v_1, \dots, v_n$ of T_pM. We need to prove that there exists a positive form on $U = M$.

Observe that if $\omega_1, \dots, \omega_m$ are positive forms on U and $\varphi_1, \dots \varphi_m\colon U \to \mathbb{R}$ are smooth functions such that $\varphi_i \geqslant 0$ on U and $\sum_i \varphi_i > 0$ on U then $\sum_i \varphi_i\omega_i$ is positive on U.

For any $p \in M$ let U_p be an open set containing p such that there exist n smooth vector fields $X_1, \dots X_n$ on U_p such that $X_1(q), \dots X_n(q)$ is a positive basis of T_qM for any $q \in U_p$. Let $X^1(q), \dots, X^n(q)$ be the dual basis of T_q^*M. Then $X^1, \dots, X^n$ are smooth forms on U (why?). Let $\omega_p = X^1 \wedge \dots \wedge X^n$. Then it's a smooth positive form on U. Take a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{U_p\}_{p \in M}$ of M. Then $\operatorname{supp} \varphi_i \subset U_{p_i}$ for some p_i and by the observation above $\omega = \sum_i \varphi_i \omega_i$ is positive on all of M. $\square$