1. Mayer-Vietoris sequence

Definition 1.0.1. A *cochain complex* K is a sequence of homomorphisms of abelian groups $\dots \xrightarrow{d_{i-1}} K^i \xrightarrow{d_i} K^{i+1} \xrightarrow{d_{i+1}} \dots$ satisfying the condition $d_{i+1} \circ d_i = 0$ for every i which we will abbreviate as $d^2 = d \circ d = 0$.

Given a cochain complex we can compute its cohomology groups

$$H^i(K) \stackrel{def}{:=} \ker d_i / \operatorname{Im} d_{i-1}.$$

Definition 1.0.2. Let K, L be two cochain complexes. A morphism $f\colon K \to L$ is a sequence of homomorphisms $f_i\colon K^i \to L^i$ satisfying $d_i \circ f_i = f_{i+1} \circ d_i$. A morphism $f\colon K \to L$ induces homomorphisms $f_*\colon H^i(K) \to H^i(L)$.

Lemma 1.0.3 (Snake Lemma)**.** *Let $0 \to K \xrightarrow{f} L \xrightarrow{g} P \to 0$ be a short exact sequence of cochain complexes. It induces a long exact sequence on cohomology*

$$\dots \to H^i(K) \xrightarrow{f_*} H^i(L) \xrightarrow{g_*} H^i(P) \xrightarrow{\delta} H^{i+1}(K) \to \dots$$

Theorem 1.0.4 (Mayer-Vietoris sequence)**.** *Let M^n be a manifold and let $U, V \subset M$ be open subsets. Then there is a long exact sequence $\dots \to H^i(M) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \to H^{i+1}(M) \to \dots$ where we put $H^i(X) = 0$ for any X and any $i < 0$.*

Proof. Let $K = \Omega^*(M), L = \Omega^*(U) \oplus \Omega^*(V), P = \Omega^*(U \cap V)$ and consider the maps $f\colon K \to L, g\colon L \to P$ given by $f(\omega) = (\omega|_U, \omega|_V)$, $g(\omega, \eta) = \omega|_{U \cap V} - \eta|_{U \cap V}$. Then $0 \to K \xrightarrow{f} L \xrightarrow{g} P \to 0$ is a short exact sequence of cochain complexes.

Exactness at K and L is straightforward. Let us verify that the sequence is exact at P, i.e. that g is onto. Using partition of unity of can construct nonnegative functions φ, ψ on M such that $\varphi + \psi \equiv 1$ and $\operatorname{supp} \varphi \subset U, \operatorname{supp} \psi \subset V$. let $\alpha \in \Omega^*(U \cap V)$. Since $\operatorname{supp}(\varphi \cdot \alpha) \subset U$ we can extend $\varphi \cdot \alpha$ by 0 to a smooth form η on V. Likewise, we can extend $\psi \cdot \alpha$ to ω on U. Then by construction $g(\omega, -\eta) = \alpha$. Thus, $0 \to K \to L \to P \to 0$ is exact. Applying the Snake Lemma we obtain the result. □

Corollary 1.0.5. *For $n \geqslant 1$ we have*

$$H^i(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} \text{ if } i = 0, n \\ 0 \text{ if } i \neq 0, n \end{cases}$$

Proof. The case of $n = 1$ was proved earlier. The general case follows by induction using Mayer-Vietoris with $M = \mathbb{S}^n, U = M \backslash P, V = M \backslash Q$ where P, Q are the north and the south pole respectively. □

2. Cohomology with compact support

Let M be a smooth manifold. Let $\Omega^*_c(M) \subset \Omega^*(M)$ be the subcomplex of forms with compact support. It's obvious that if $\omega \in \Omega^*_c(M)$ then

$d\omega \in \Omega^*_c(M)$ as well. Therefore we can define the cohomology of $\Omega^*_c(M)$ which we will denote by $H^*_c(M)$.

Example 2.0.1.

- If M is compact then $\Omega^*_c(M) = \Omega^*(M)$ and therefore $H^*_c(M) = H^*(M)$.
- Let M be connected and noncompact. Then $H^0(M) = 0$. Indeed. let $\omega = f$ be a closed 0-form with compact support. Since M is connected, $df = 0$ means that $f \equiv$const and since f has compact support and M is noncompact this means that $f \equiv 0$.
- Let M^n be connected and oriented manifold without boundary. Then for any $\eta \in \Omega^{n-1}_c(M)$ we have that $\int_M d\eta$ is defined and by Stokes's formula $\int_M d\eta = 0$. This means that the map $\omega \mapsto \int_M \omega$ induces a well-defined homomorphism $\int\colon H^n_c(M) \to \mathbb{R}$. It's obvious that this map is onto.
- Let $M = \mathbb{R}$. By above $H^0_c(\mathbb{R}) = 0$. By degree reasons $H^i_c(\mathbb{R}) = 0$ for $i > 1$. We claim that $H^1_c(\mathbb{R}) \cong \mathbb{R}$. By above it's enough to show that the map $\int\colon H^n_c(M) \to \mathbb{R}$ is injective. Let $\omega \in \Omega^1_c(\mathbb{R})$ be such that $\int_{\mathbb{R}} \omega = 0$. we need to show that $\omega = df$ for some compactly supported f. We have $\omega = \varphi(t)dt$ for some $\varphi \in C^\infty_c(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \varphi(t)dt = 0$. Define $f(x) = \int_{-\infty}^{x} \varphi(t)dt$. Then $df = \varphi(t)dt$ and since $\int \varphi = 0$ it follows that f has compact support.

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a bump function with compact support such that $\int_{\mathbb{R}} \varphi = 1$. Let $e = \varphi(t)dt$ be a generator of $H^1_c(\mathbb{R})$. Consider the following maps

$$e_*\colon \Omega^*_c(\mathbb{R}^n) \to \Omega^{*+1}_c(\mathbb{R}^n \times \mathbb{R}) \text{ given by } \omega \mapsto \omega \wedge e$$

$$\pi_*\colon \Omega^{*+1}_c(\mathbb{R}^n \times \mathbb{R}) \to \Omega^*_c(\mathbb{R}^n)$$

given by

$$\omega = \sum_{|I|=*+1} \omega_I(x,t)dx^I + \sum_{|J|=*} \omega_J(x,t)dx^J \wedge dt \mapsto \sum_{|J|=*} (\int_{-\infty}^{+\infty} \omega_J(x,t)dt)dx^J$$

One can check that $d \circ \pi_* = \pi_* \circ d$ and $d \circ \epsilon_* = e_* \circ d$ (Exercise: verify this!). Thus π_* and e_* induce maps on compactly supported cohomology $\pi_*\colon H^{*+1}_c(\mathbb{R}^n \times \mathbb{R}) \to H^*_c(\mathbb{R}^n)$ and $e_*\colon H^*_c(\mathbb{R}^n) \to H^{*+1}_c(\mathbb{R}^n \times \mathbb{R})$.

It's easy to see that $\pi_* \circ e_*(\omega) = \omega$ for any $\omega \in \Omega^*_c(\mathbb{R}^n)$. Therefore $\pi_* \circ e_* = \mathrm{id}$ on $H^*_c(\mathbb{R}^n)$.

Lemma 2.0.2 (Poincare lemma with compact support)**.**

$$e_* \circ \pi_* = \mathrm{id} \ \ \textit{on } H^{*+1}_c(\mathbb{R}^n \times \mathbb{R})$$

and hence π_* *and* e_* *are inverse isomorphisms.*

Proof. Let $A(t)=\int_{-\infty}^{t}\varphi(s)ds$. Define a "homotopy operator" $K\colon\ \Omega_c^{*+1}(\mathbb{R}^n\times\mathbb{R})\to\Omega_c^{*}(\mathbb{R}^n\times\mathbb{R})$ by the formula

$$\omega=\sum_{|I|=*+1}\omega_I(x,t)dx^I+\sum_{|J|=*}\omega_J(x,t)dx^J\wedge dt\overset{K}{\mapsto}$$

$$\sum_{|J|=*}[(\int_{-\infty}^{t}\omega_J(x,s)ds)dx^J-A(t)(\int_{-\infty}^{+\infty}\omega_J(x,s)ds)dx^J]$$

Claim: For any $\omega\in\Omega_c^{k+1}(\mathbb{R}^n\times\mathbb{R})$ it holds

$$e_*\circ\pi_*(\omega)-\omega=(-1)^k(dK-Kd)\omega$$

The proof is a direct calculation. Once this formula is established it immediately follows that if $d\omega=0$ then $e_*\circ\pi_*(\omega)-\omega$ is exact which proves the Lemma.

□

By induction on n the Lemma immediately gives

Corollary 2.0.3.

$$H_c^*(\mathbb{R}^n)\cong\begin{cases}\mathbb{R}\ \textit{if}\ *=n\\ 0\ \textit{if}\ *\neq n\end{cases}$$

We already know that the map $\int\colon\ H_c^n(\mathbb{R}^n)\to\mathbb{R}$ is onto. Since by the above corollary the domain is 1-dimensional this implies

Corollary 2.0.4. *The map $\int\colon\ H_c^n(\mathbb{R}^n)\to\mathbb{R}$ is an isomorphism. In particular $w_1,w_2\in\Omega_c^n(\mathbb{R}^n)$ are cohomologous if and only if $\int_{\mathbb{R}^n}\omega_1=\int_{\mathbb{R}^n}\omega_2$.*

Theorem 2.0.5. *Let M^n be a connected oriented manifold without boundary. Then the map $\int\colon\ H_c^n(M)\to\mathbb{R}$ is an isomorphism.*

Proof. We can cover M by a countable collection of positive charts $x_i\colon\ U_i\to\mathbb{R}^n$. Let $\omega\in\Omega_c^n(M)$ is such that $\int_M\omega=0$. We need to show that $\omega=d\eta$ for some $\eta\in\Omega_c^{n-1}(M)$. Since $\operatorname{supp}(\omega)$ is compact it's contained in the union of finitely many U_is. Since M is connected by adding a few more U_i's and rearranging the terms we can assume that $\operatorname{supp}\omega\subset\cup_{i=1}^N U_i$ and that $M_k=\cup_{i=1}^k U_i$ is connected for any $i\leqslant N$.

We claim that $H_c^n(M_k)\cong\mathbb{R}$ for any $k\leqslant N$. We proceed by induction in k. For $k=1$ the claim is true by Corollary 2.0.4.

Suppose the claim has been established for k. We need to verify that it holds for $k+1$. Let $\eta\in\Omega_c^n(M_{k+1})$ such that $\int_M\eta=0$. It's enough to show that $\eta=d\beta$ for some $\beta\in\Omega_c^{n-1}(M_{k+1})$. Using partition of unity we can construct nonnegative smooth functions φ,ψ on M_{k+1} such that $\varphi+\psi\equiv1$, $\operatorname{supp}\varphi\subset M_k$ and $\operatorname{supp}\psi\subset U_{k+1}\cong\mathbb{R}^n$. Let $\alpha\in\Omega_c^n(M_k\cap U_{k+1})$ be an axillary form with $\int_M\alpha=1$. Let $\eta_1=\varphi\eta,\eta_2=\psi\eta$ and let $c=\int_M\eta_1$. Then $\eta=\eta_1+\eta_2$, $0=\int_M\eta=\int_M\eta_1+\int_M\eta_2$ and hence $\int_M\eta_2=-c$.

Consider the form $\eta_1 - c\alpha$. It has support contained in M_k and $\int_M(\eta_1 - c\alpha) = c - c = 0$. hence by the induction assumption, $\eta_1 = d\beta_1$ for some $\beta_1 \in \Omega_c^{n-1}(M_k)$. Similarly, the form $\eta_2 + c\alpha$ has support in U_{k+1} and $\int_M \eta_2 + c\alpha = -c + c = 0$ also. Therefore, $\eta_2 = d\beta_2$ for some $\beta_2 \in \Omega_c^{n-1}(U_{k+1})$. Thus, $\eta = \eta_1 + \eta_2 = d(\beta_1 + \beta_2)$. This finishes the proof of the induction step and hence of Theorem 2.0.5. □

Since for compact manifolds $H^n = H_c^n$, the above theorem immediately yields

Corollary 2.0.6. *Let M^n be a closed connected orientable manifold. Then $H^n(M) \cong \mathbb{R}$.*

3. Degree theory

Definition 3.0.1. Let M^n, N^n be closed connected oriented manifolds. Let $f\colon M \to N$ be a smooth map. Let $c \in N$ be a regular value of f. By the Inverse Function Theorem $f^{-1}(c)$ is a compact submanifold of M of dimension 0, i.e. it's a finite set of points. Let $f^{-1}(c) = \{p_1, \dots, p_k\}$. Let $\operatorname{sign}(p_i) = +1$ or -1 depending on whether or not df_{p_i} is orientation preserving. Define the degree of f by the formula

$$\deg f = \sum_i \operatorname{sign}(p_i)$$

Theorem 3.0.2. *Degree is well defined, i.e. it does not depend on the choice of a regular value c. Moreover, for any $\omega \in \Omega^n(N)$ it holds that*

$$\int_M f^*\omega = \deg f \cdot \int_N \omega \tag{3.0.1}$$

Proof. It's obviously sufficient to prove formula (3.0.1) as it implies that degree of f is invariantly defined. By the Inverse Function Theorem we can find open sets U_i containing p_i and and open set U containing p such that U is diffeomorphic to $\mathbb{R}^n$ and $f|_{U_i}\colon U_i \to U$ is a diffeomorphism for every $i = 1, \dots, k$.

Let $\alpha \in \Omega_c^n(U)$ be such that $\int_N \alpha = 1$. Let $c = \int_N \omega$. Since $H^n(N) \cong \mathbb{R}$ we have that $[\omega] = [c\alpha] \in H^n(N)$.

We have $\int_M f^*(\alpha) = \sum_i \int_{U_i} f^*(\alpha) = \sum_i \operatorname{sign}(p_i) \int_U \alpha = (\deg f) \int_U \alpha = \deg f$. Since $[\omega] = [c\alpha]$ we have that $[f^*(\omega)] = [f^*(c\alpha)] = [cf^*(\alpha)]$. Therefore,

$$\int_M f^*(\omega) = \int_M cf^*(\alpha) = c\int_M f^*(\alpha) = c \cdot \deg f = \deg f \cdot \int_N \omega$$

□

Corollary 3.0.3 (Homotopy invariance of degree)**.** *Let $f, g\colon M \to N$ be homotopic. Then*

$$\deg f = \deg g$$

Example 3.0.4. Let $A\colon \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map $A(x) = -x$. Then $\deg A = (-1)^{n+1}$.

Example 3.0.5. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be given by $f(z) = z^n$. Then $\deg f = n$.

Corollary 3.0.6. *Supose $f\colon M \to N$ has $\deg f \neq 0$. Then f is onto.*

Proof. Suppose f is not onto. Let $c \in N$ be a point which is not in the image of f. Then c is a regular value of f. using ing for computing $\deg f$ gives that $\deg f = 0$. □

Theorem 3.0.7 (Fundamental Theorem of Algebra)**.** *Let $n \geqslant 1$ and let $p(z) = a_n z^n + \ldots a_1 z + a_0$ be a complex polynomial of degree n with $a_i \in C$ and $a_n \neq 0$.*

Then there exists $z \in \mathbb{C}$ such that $p(z) = 0$.

Proof. Consider the following map $f_p\colon \mathbb{CP}^1 \to \mathbb{CP}^1$

$$f_p([z_0 : z_1]) = [a_n z_0^n + a_{n-1} z_0^{n-1} z_1 + \ldots + a_1 z_0 z_1^{n-1} + a_0 z_1^n : z_1^n]$$

This map is easily seen to be smooth. Moreover, when written with respect to the standard parameterization $\varphi(u) = [u : 1]$ and the inverse coordinate chart $x([z_0 : z_1]) = \frac{z_0}{z_1}$ defined on the open set $U = \{[z_0 : z_1] : z_1 \neq 0\}$ we have that $x \circ f_p \circ \varphi = p$. Also, obviously, $f_p([1 : 0]) = [a_n : 0] = [1 : 0]$.

Let $p_t(z) = a_n z^n + t a_{n-1} z^{n-1} + \ldots t a_1 z + t a_0$.

Then $p_0(z) = a_n z^n$ and $p_1 = p$. The map $F(t, [z_0 : z_1]) = f_{p_t}([z_0 : z_1])$ provides a homotopy between f_p and f_{p_0}. Since $\mathbb{C}\backslash\{0\}$ is connected, it's easy to see that $f_{a_n z^n}$ is homotopic to f_{z^n}. Therefore, $\deg f_p = \deg f_{a_n z^n} = f_{z^n}$. By taking $c = 1$ and looking at the roots of $z^n = 1$ it's easy to compute that $\deg f_{z^n} = n$. Thus, $\deg f_p = n \neq 0$. Therefore, by corollary 3.0.6, f_p is onto. In particular, there exists $[z_0 : z_1]$ such that $f_p([z_0 : z_1]) = [0 : 1]$. By above, $z_1 \neq 0$ and therefore $p(\frac{z_0}{z_1}) = 0$. □

Lemma 3.0.8. *Let $f, g\colon X \to S^n$ satisfy $f(x) \neq -g(x)$ for any $x \in X$. Then $f \sim g$.*

Proof. Let $F\colon X \times [0,1] \to S^n$ be given by $F(x,t) = \frac{tf(x) + (1-t)g(x)}{|tf(x) + (1-t)g(x)|}$. Since $f(x) \neq -g(x)$ the denominator is never zero. Hence F is a homotopy from f to g. □

Theorem 3.0.9 (Hairy Ball Theorem)**.** *There is no continuous nowhere vanishing vector field V on $\mathbb{S}^{2n}$ for any $n \geqslant 1$.*

Proof. Suppose such V exists. By changing V to $\frac{V}{|V|}$ we can assume that $|V(x)| = 1$ for any $x \in \mathbb{S}^{2n}$. Thus V can be viewed as a map $V\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$. Since $V(x) \in T_x \mathbb{S}^{2n}$ we have that $V(x) \perp x$ for any x. Applying the previous lemma to V and the identity map of $\mathbb{S}^{2n}$ we conclude that $V \sim \mathrm{id}_{\mathbb{S}^{2n}}$. Similarly, $V \sim A$ where $A\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ is the antipodal map $A(x) = -x$. Thus $\mathrm{id}_{\mathbb{S}^{2n}} \sim V \sim A$. This is a contradiction since $\deg \mathrm{id}_{\mathbb{S}^{2n}} = 1$ and $\deg A = -1$. □

4. Euler Characteristic

Let M be a closed oriented manifold. Let V be a vector field on M transverse to the zero section. Let p be a zero of V. Define $\operatorname{sign}(p)$ as follows. let x be a positive local coordinates near p. Then V can be written as $V = \sum_i V_i(x)\frac{\partial}{\partial x_i}$. Transversality means that $\det\left(\frac{\partial V_i}{\partial x_j}(p)\right) \neq 0$. Define $\operatorname{sign}(p) := \operatorname{sign}(\det\left(\frac{\partial V_i}{\partial x_j}(p)\right))$. It's easy to see that this definition does not depend on the choice of the positive chart x.

Definition 4.0.1. Let M be a closed oriented manifold.

Let V be a smooth vector field on M such that $V \pitchfork 0(M)$. Let $p_1, \dots, p_k$ be the zeros of V. Set

$$\chi_V(M) = \sum_i \operatorname{sign}(p_i)$$

Theorem 4.0.2. *Then $\chi_V(M)$ does not depend on the choice of the transverse vector field V.*

Sketch of the Proof. Let V_0, V_1 be two different smooth vector fields on M which are transverse to the zero section. We need to show that $\chi_{V_0}(M) = \chi_{V_1}(M)$.

Let $F\colon M \times [0,1] \to TM$ be the homotopy between V_0 and V_1 given by $F(p,t) = (1-t)V_1(p) + tV_2(p)$. Then F is a smooth map and it's transverse to $0(M)$ on its boundary $\partial(M \times [0,1]) = M \times \{0\} \cup M \times \{1\}$. By the Transversality Aproximation Theorem F is homotopic to a map $\tilde{F}\colon M \times [0,1] \to TM$ such that $\tilde{F} \pitchfork 0(M)$ and $\tilde{F}|_{\partial(M\times[0,1])} = F|_{\partial(M\times[0,1])}$, i.e. $\tilde{F}(p,0) = V_0(p)$ and $\tilde{F}(p,1) = V_1(p)$ for any $p \in M$.

Then the set $S = \tilde{F}^{-1}(0(M)) \subset M \times [0,1]$ is a compact submanifold with boundary of dimension 1 and $\partial S \subset \partial(M \times [0,1]) = M \times \{0\} \cup M \times \{1\}$. By construction, $\partial S \cap M \times \{0\} = V_0^{-1}(0)$ and $\partial S \cap M \times \{1\} = V_1^{-1}(0)$.

Observe that S inherits a natural orientation defined as follows. Let $p \in S, v \in T_pS$. We have that $F(p) = (q,0)$ for some $q \in M$. Let $x\colon U \to \mathbb{R}^n$ be positive local coordinates near q and let (x,y) be corresponding standard local coordinates on TU. Then F can be written as $F(z) = \sum_i y_i(z)\frac{\partial}{\partial x_i}|_{x(z)}$. Let $G\colon TU \to \mathbb{R}^n$ be given by $G(x,y) = y$ and let $Y = G \circ F$. Note that $G(p) = 0$. The condition that F is transverse to the zero section at p is equivalent to saying that $dY_p\colon T_p(M \times [0,1]) \to \mathbb{R}^n$ is onto. Let $v_2, \dots v_n \in T_p(M \times [0,1])$ be such that $dY_p(v_2), \dots, dY_p(v_n)$ is a positive basis of $\mathbb{R}^n$.

For $v \in T_pS$ we say that it's positively oriented if $V, v_2, \dots, v_n$ is a positive basis of $T_p(M \times [0,1])$. It's easy to see that this orientation does not depend on the choices involved.

Since S is compact and 1-dimensional it's diffeomorphic to a disjoint union of finitely many circles and closed intervals.

One can check that for any zero of $(p_i, 1)$ of V_1 it holds that $\operatorname{sign}(p_i)$ is equal to the induced orientation on $(p_i, 1)$ when viewed as a point on the boundary of S. Likewise, for any zero $(q_i, 1)$ of V_q it holds that $\mathrm{sign}(q_i)$ is equal to the induced orientation on $(q_i, 1)$ when it is viewed as a point on the boundary of S.

Thus the arcs in S with both endpoints on $M \times \{1\}$ contribute one $+1$ and one -1 to $\chi_{V_1}(M)$ and the same works for the arcs in S with both boundary points on $M \times \{0\}$. Thus, the only components of S that nontrivially contribute to $\chi_{V_0}(M)$ and $\chi_{V_1}(M)$ are those with one endpoints on $M \times \{1\}$ and another on $M \times \{0\}$. This yields that $\chi_{V_0}(M) = \chi_{V_1}(M)$. □

In view of Theorem 4.0.2 We can define the Euler characteristic of M to be

$$\chi(M) := \chi_V(M)$$

where V is any vector field on M transverse to the zero section.

Note that if M admits a nowhere vanishing vector field on V. Then $\chi(M) = \chi_V(M) = 0$.

Example 4.0.3. Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$ be the unite sphere centered at 0 and let $V(x, y, z) = (-y, x, 0)$. Then $V \pitchfork 0(M)$ with the zeros at $(0, 0, \pm 1)$. One can check that both zeros have signs=+1 and therefore $\chi(\mathbb{S}^2) = 2$. This gives a different proof of the Hairy Ball Theorem.