MAT 1300F Topology I
Assignment 8 Solutions
Nov. 24, 2015

2. Note that $H^1_{DR}(\mathbb{R}^2) = \dfrac{\ker d \cap \Omega^1}{imd \cap \Omega^1}$. Therefore, it suffices to show that any closed 1-form $\omega \in \ker d$ is exact.

Suppose that $\omega = \alpha dx + \beta dy$ is closed, where $\alpha, \beta \in C^\infty(\mathbb{R}^2)$.

Then
$$d\omega = \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx \wedge dy = 0 \Rightarrow \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} = 0.$$

Let $f = \int_0^x \alpha(s,0)ds + \int_0^y \beta(x,t)dt$. We show that $df = \omega$. Indeed,

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
&= \frac{\partial}{\partial x}\left(\int_0^x \alpha(s,0)ds + \int_0^y \beta(x,t)dt\right) dx + \frac{\partial}{\partial y}\left(\int_0^y \beta(x,t)dt\right) dy \\
&= \left(\alpha(x,0) + \int_0^y \frac{\partial \beta}{\partial x}dt\right) dx + \beta(x,y)dy \\
&= \left(\alpha(x,0) + \int_0^y \frac{\partial \alpha}{\partial y}dt\right) dx + \beta(x,y)dy \\
&= \alpha dx + \beta dy = \omega.
\end{aligned}$$

14-6(a) Because $(x,y,z) = F(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$, we have

$$\begin{aligned}
F^*(dx) &= \sin\varphi\cos\theta d\rho + \rho\cos\varphi\cos\theta d\varphi - \rho\sin\varphi\sin\theta d\theta \\
F^*(dy) &= \sin\varphi\sin\theta d\rho + \rho\cos\varphi\sin\theta d\varphi + \rho\sin\varphi\cos\theta d\theta \\
F^*(dz) &= \cos\varphi d\rho - \rho\sin\varphi d\varphi.
\end{aligned}$$

Therefore, $F^*(\omega) = F^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = \rho^3 \sin\varphi d\varphi \wedge d\theta$.

14-6(b) From above, in spherical coordinates, $d\omega = 3\rho^2 \sin\varphi d\rho \wedge d\varphi \wedge d\theta$.

Note that $d\omega = 3dx \wedge dy \wedge dz$. Then

$$\begin{aligned}
F^*(d\omega) = 3F^*(dx \wedge dy \wedge dz) &= 3\det\left(\frac{\partial F_i}{\partial x^j}\right) d\rho \wedge d\varphi \wedge d\theta \\
&= 3\rho^2 \sin\varphi d\rho \wedge d\varphi \wedge d\theta.
\end{aligned}$$

14-6(c) In the coordinate chart (φ,θ) on S^2, $\iota^*(\omega) = \sin\varphi d\varphi \wedge d\theta$.

14-6(d) From above, on $S^2 - \{N = (0,0,1), S = (0,0,-1)\}$, because $\sin\varphi \neq 0$, we have that $\iota^*(\omega)$ is nonzero.

To see that $\iota^*(\omega)$ is non-vanishing on the entire sphere, we need to check at N, S. Let $e_1 = (1,0,0), e_2 = (0,1,0) \in T_N(S^2) \subset T_N(R^3)$ (resp. $T_S(S^2)$).

At $N = (0, 0, 1)$,

$$\iota^*(\omega)(e_1, e_2) = \omega(d\iota(e_1), d\iota(e_2)) = dx \wedge dy(e_1, e_2) = 1 \neq 0.$$

Similarly, at $S = (0, 0, -1)$,

$$\iota^*(\omega)(e_1, e_2) = -1 \neq 0.$$

Therefore, we conclude that $\iota^*(\omega)$ is a non-vanishing form on S^2.