Solutions to selected problems from homework 7

(1) Let V be a smooth vector field on a smooth manifold M without boundary. Let $p \in M$ and suppose $V(p) \neq 0$.

Prove that there exist local coordinates x on an open set U containing p such that in these coordinates $V \equiv \frac{\partial}{\partial x_1}$.

Hint: Use an argument similar to the proof of the collar neighbourhood theorem.

Solution

Since the question is local we can assume that $M = U$ is an open set in $\mathbb{R}^n$ and $p = 0$. Let $N = V(p)^\perp \cong \mathbb{R}^{n-1}$. Let $\Phi\colon (-\varepsilon, \varepsilon) \times N \to \mathbb{R}^n$ be the the integral flow of V restricted to N.

Since $\phi_0(x) = x$ for any x we have that $d\Phi_{(0,p)}(0, v) = v$ for any $v \in N$. Also, by definition of the flow we have that $d\Phi_{(0,p)}(\frac{\partial}{\partial t}, 0) = V(p)$. Since $V(p) \notin N$ and $\dim M = \dim(\mathbb{R} \times N) = n$ this implies that $d\Phi_{(0,p)}$ is an isomorphism. Therefore, by the Inverse Function Theorem, there is an open neighbourhood $U_p \subset N$ containing p and $\varepsilon_1 > 0$ such that $\phi|_{(-\varepsilon_1, \varepsilon_1) \times U_p}$ is a diffeomorphism onto its image which is an open neighbourhood of p in M. By construction $x = \Phi^{-1}$ has the desired properties.

(2) Let V be a vector space of dimension n. An alternating k-tensor ω on V is called *decomposable* if it can be written as $\omega = \eta \wedge \nu$ where η and ν have degrees smaller than ω.

(a) Let $V = \mathbb{R}^4$ and $\omega = e^{12} + e^{34}$. Prove that ω is not decomposable.

(b) Let $V = \mathbb{R}^n$ where $n \geq 4$ and $\omega = e^{12} + e^{34}$. Prove that ω is not decomposable.

Hint: Given $\omega \in \Lambda^{n-2}(\mathbb{R}^n)$ consider the map $L_\omega\colon \Lambda^1(\mathbb{R}^n) \to \Lambda^{n-1}(\mathbb{R}^n)$ given by $L_\omega(\eta) = \omega \wedge \eta$. Look at the dimension of $\ker L_\omega$.

Solution

Let us prove part (a) first. Let $V = \mathbb{R}^4$.

Consider the map $L_\omega\colon \Lambda^1(V^*) \to \Lambda^3(V^*)$ given by $L_\omega(\eta) = \omega \wedge \eta$.

We know that e^1, e^2, e^3, e^4 is a basis of $\Lambda^1(V^*)$ and $e^{234}, e^{134}, e^{124}, e^{123}$ is a basis $\Lambda^3(V^*)$. Direct computation shows that

$L_\omega(e^1) = (e^{12} + e^{34}) \wedge e^1 = 0 + e^{341} = e^{134}, L_\omega(e^2) = e^{234}, L_\omega(e^3) = e^{123}, L_\omega(e^4) = e^{124}$. This means that L_ω is onto and since $\dim \Lambda^1(V^*) = \Lambda^3(V^*) = 4$, L_ω is an isomorphism. In particular, $\ker L_\omega = 0$.

Now suppose ω is decomposable. Then it can be written as $\omega = \omega_1 \wedge \omega_2$ for some nonzero (in fact, linearly independent) $\omega_1, \omega_2 \in \Lambda^1(V^*)$. Then $L_\omega(\omega_1) = \omega_1 \wedge \omega_1 \wedge \omega_2$. Hence $\ker L_\omega \neq 0$.

This is a contradiction and hence ω is not decomposable which proves part (a).

Let us now show that the same ω is not decomposable for $V = \mathbb{R}^n$ for $n > 4$.

Let $\tilde{\omega} = \omega \wedge e^{56\ldots n} = e^{1256\ldots n} + e^{3456\ldots n} \neq 0$. Consider the map $L_{\tilde{\omega}} \colon \Lambda^1(V^*) \to \Lambda^{n-1}(V^*)$ given by $L_{\tilde{\omega}}(\eta) = \tilde{\omega} \wedge \eta$. As before, it's easy to see that image of $L_{\tilde{\omega}}$ has dimension 4 and hence $\dim\ker L_{\tilde{\omega}} = n-4$. Now suppose that ω is decomposable and $\omega = \omega_1 \wedge \omega_2$. Then arguing as before it's easy to see that $\dim\ker L_{\tilde{\omega}} > n-4$. This is a contradiction and hence ω is indecomposable. $\square$

(3) Let V, W be finite-dimensional vector spaces and let $f \colon V \to W$ be a linear map. Using the definition of wedge product given in class prove that $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ for any alternating tensors ω, η on W.

Solution

By linearity it's enough to prove that for any $I = (i_1, \ldots, i_k)$ it holds that $f^*(e^I) = f^*(e^{i_1}) \wedge \ldots \wedge f^*(e^{i_k})$. Let's evaluate both sides of this formula on $v_1, \ldots, v_k$.

LHS gives $f^*(e^I)(v_1, \ldots, v_k) = e^I(f(v_1), \ldots, f(v_k)) = \det(e^{i_s}(f(v_{j_t})))$. Evaluating the RHS we get

$$f^*(e^{i_1}) \wedge \ldots \wedge f^*(e^{i_k})(v_1, \ldots, v_k) = \det(f^*(e^{i_s})(v_{j_t})) = \det(e^{i_s}(f(v_{j_t}))).$$

Thus, $LHS = RHS$. Since $v_1, \ldots, v_k$ are arbitrary this proves that $f^*(e^I) = f^*(e^{i_1}) \wedge \ldots \wedge f^*(e^{i_k})$. $\square$.

(4) Let $U \subset \mathbb{R}^n$ be open and let $V_1, \ldots V_n$ be smooth vector fields on U such that for any $x \in U$, $V_1(x), \ldots V_n(x)$ is a basis of $\mathbb{R}^n$.

Let $w_1, \ldots w_n$ be the dual collection of 1 forms, i.e. for any $x \in U$ $w_1(x), \ldots w_n(x)$ is the unique n-tuple of elements of $(\mathbb{R}^n)^*$ satisfying $w_i(x)(V_j(x)) = \delta_{ij}$.

Prove that all w_i are smooth.

Solution

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. A 1-form α on U can be written as $\alpha = \sum_i \alpha_i(x) dx^i$ where $\alpha_i(x) = \alpha(x)(e_i)$. Then α is smooth iff $\alpha_i(x)$ is smooth for every i.

Thus, we need to check that $\omega_j(x)(e_i)$ is smooth in x for every i, j. Let $A(x) = [V_1(x), \ldots, V_n(x)]$. Then $A(x)$ is an invertible $n \times n$ matrix. By assumption it depends smoothly on x.

We have $(V_1(x), \ldots, V_n(x)) = (e_1, \ldots, e_n) \cdot A(x)$. Therefore, $(e_1, \ldots, e_n) = (V_1(x), \ldots, V_n(x)) \cdot A^{-1}(x)$. Observe that $A^{-1}(x)$ is also smooth in x because by the general formula for the inverse of a matrix, A^{-1} has entries which are rational functions in coefficients of A.

Thus, $e_i = \sum_k A^{-1}_{ki}(x) V_k(x)$. Therefore, $\omega_j(e_i)(x) = \sum_k A^{-1}_{ki}(x) \omega_j(V_k)(x) = \sum_k A^{-1}_{ki}(x) \delta_{jk} = A^{-1}_{ji}(x)$ is smooth in x. $\square$.