Solutions to selected problems from homework 5

(1) Recall that given a smooth manifold with boundary M^n we call a set $S \subset M$ a submanifold (with boundary) of M of dimension k if for every $p \in S$ there exist an open set U containing p and a diffeomorphism $x\colon U \to V$ where V is an open set in H^n such that $x(U \cap S) = V \cap H^k$ for some $H^k \subset H^n$.

Let $M = H^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and let $S = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$. Prove that S is not a submanifold in H^2 but it is a submanifold in $\mathbb{R}^2$.

(2) Let M be a manifold with boundary and let N be a manifold without boundary. Let $f\colon M \to N$ be smooth and let $c \in N$ be a regular value for both f and $f|_{\partial M}$.

Prove that $S = f^{-1}(c)$ is a smooth submanifold with boundary in M and $\partial S \subset \partial M$.

Solution

By the corresponding result for manifolds without boundary we already know that $S \cap int(M)$ is a k-dimensional submanifold on $int(M)$ without boundary. Thus, we only need to analyze the what S looks like near $p \in S \cap \partial M$.

First observe the following. Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be smooth. Let $x = (x_1, \ldots x_k), y = (y_1, \ldots, y_m)$ be standard coordinates on $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively. Let $n = k + m$

Let T be the graph of f over the half-space $x_1 \geq 0$. I.e. let $T = \{(x, f(x)) : x_1 \geq 0\}$. Then T is a smooth k-dimensional submanifold in the half-space $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^n : x_k \geq 0\}$. Indeed, the map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $F(x, y) = (x, y - f(x))$ is a diffeomorphism (with inverse $(x, y) \mapsto (x, y + f(x))$ and $F(T) = \{(x, 0); x_1 \geq 0$.

We will show that S has the above form locally near $p \in S \cap \partial M$. By considering appropriate local coordinates near p we can assume that $M = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_1 \geq 0\}$, $p = 0$ and $N = \mathbb{R}^m$.

Since c is a regular value of f we have that df_p has maximal rank, i.e. the $n \times m$ matix

$$[df_p] = \left(\frac{\partial f_i}{\partial x_j}(p)\right)$$

has rank m. Therefore it has m linearly independent columns $\frac{\partial f}{\partial x_{j_1}}(p), \ldots, \frac{\partial f}{\partial x_{j_m}}(p)$.

Since c is a regular value for $f|_{\partial M}$ we know that in fact the same holds true for the function $f(0, x_2, \ldots, x_n)$ i.e. we can assume that none of the indices $j_1, \ldots, j_m$ are equal to 1. Thus, after possibly relabelling coordinates from 2 to n we can assume that the last m columns in $[df]_p$ are linearly independent. By the inverse function theorem that means that near p the level set $\{f = c\}$ has the form

$\{(x_1, \ldots, x_k, h(x_1, \ldots, x_k)\} : x - 1 \geq 0\}$ for some smooth h which is a submanifold by the observation above.

(3) Let $U, V \subset H^n$ be open sets containing $p = 0$ and let $f \colon U \to V$ be a diffeomorphism such that $f(p) = p$. Let $v = (v_1, \ldots, v_n)$ be a vector in $\mathbb{R}^n$ with $v_n > 0$.

Let $df_p(v) = w = (w_1, \ldots, w_n)$.

Prove that $w_n > 0$.

Solution

Let $\gamma(t) = tv$. Then by the chain rule, $df_p(v) = (f \circ \gamma)'(0)$. Since $f(0) = 0$ and $f(\gamma(t)) \in H^n$ for any $t > 0$ we must have that $(f \circ \gamma)'(0) \in H^n$. Therefore $w_n \geq 0$. To see that this inequality must be strict recall that it was proved in class that $f|_{\partial H^n}$ is a diffeomorphism from $\partial H^n \cap U$ to $\partial H^n \cap V$ which implies that df_p gives an isomorphism of $\mathbb{R}^{n-1} \times \{0\}$ to itself and the same is true for its inverse. Therefore df_p can not sent vectors from $\mathbb{R}^n \backslash (\mathbb{R}^{n-1} \times \{0\})$ to $\mathbb{R}^{n-1} \times \{0\}$ and hence we can not have $w_n = 0$.