Solutions to selected problems from homework 3

(1) Problem 2-14 from the book:

Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for any $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Solution

By Theorem 2.29 from the book there exist smooth nonnegative functions f_A, f_B on M such that $f_A^{-1}(0) = A$ and $f_B^{-1}(1) = B$. It's immediate to check that $f(x) = \frac{f_A(x)}{f_A(x)+f_B(x)}$ satisfies the required properties.

(2) Prove that there exists a diffeomorphism $f\colon [0,1) \to [0,\infty)$ such that $f(x) = x$ for small x.

Solution

Let us first construct a smooth function $g\colon [0,1) \to \mathbb{R}$ such that $g(t) > 0$ for any t, $g(t) = 1$ for small t and $\int_0^1 g(t)dt = \infty$.

Let $g_1(t) = \frac{1}{1-t}$. Note that $g_1 > 0$ on $[0,1)$ and $\int_{1-\epsilon}^1 g_1(t)dt = \infty$ for any $0 < \epsilon < 1$. Pick $\epsilon < 1/2$. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative bump function centered at 1. I.e. $\phi \ge 0$, $\text{supp}(\phi) = [1-\epsilon, 1+\epsilon]$ and $\phi(t) \equiv 1$ on $[1-\epsilon/2, 1+\epsilon/2]$.

Let $g(t) = 1 + \phi(t)g_1(t)$. It's immediate to check that $g(t)$ satisfies all the required conditions.

Now set $f(x) = \int_0^x g(t)dt$. Then $f(x)$ is smooth, $f(x) = x$ for small x, $f(x)$ is strictly increasing and $\lim_{x\to 1} f(x) = \infty$. This means that $f\colon [0,1) \to [0,\infty)$ is a smooth bijection. Moreover, since $f'(x) = g(x) > 0$ for any x we have that f is a local diffeomorphism by the Inverse Function Theorem. Thus f^{-1} is also smooth and hence f is a diffeomorphism.

(3) Look at the surface of revolution M^2 in $\mathbb{R}^3$ obtained by rotating the circle of radius 1 centered at $(2,0)$ around the vertical axes.

(a) Verify that it's given by the equation

$$(\sqrt{x^2+y^2} - 2)^2 + z^2 = 1$$

and that M is a smooth manifold.

(b) Prove that M is diffeomorphic to $S^1 \times S^1$.

Solution

Note that $(0,0,0)$ does not satisfy $(\sqrt{x^2+y^2}-2)^2+z^2 = 1$. Thus to see (a) it is enough to verify that 1 is a regular value of $f\colon \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$ given by $f(x,y,z) = (\sqrt{x^2+y^2}-2)^2+z^2$. We compute $\frac{\partial}{\partial x}f(x,y,z) = 2(\sqrt{x^2+y^2}-2)\frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial y}f(x,y,z) = 2(\sqrt{x^2+y^2}-2)\frac{y}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial z}f(x,y,z) = 2z$.

Suppose $df_{(x,y,z)} = 0$ for some (x,y,z) satisfying $f(x,y,z) = 1$. Then $0 = \frac{\partial}{\partial z} f(x,y,z) = 2z$. Hence $z = 0$. Since $f(x,y,z) = 1$ this implies that $(\sqrt{x^2+y^2} - 2)^2 = 1$. Therefore $\frac{\partial}{\partial x} f(x,y,z) = 0$, $\frac{\partial}{\partial y} f(x,y,z) = 0$ imply that $x = y = 0$. Thus, $x = y = z = 0$. This is a contradiction since $(0,0,0)$ does not satisfy $(\sqrt{x^2+y^2} - 2)^2 + z^2 = 1$. Therefore, $M = \{(\sqrt{x^2+y^2} - 2)^2 + z^2 = 1\}$ is a smooth 2-dimensional manifold.

Next, consider the map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $F(\theta,\phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$. It's easy to see that $F(\mathbb{R}^2) = M$.

Clearly, $F(\theta+n, \phi+m) = F(\theta,\phi)$ for any $(m,n) \in \mathbb{Z}^2$ and hence F induces a well defined map $\bar{F}\colon T^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \to M$. It's easy to see that $\bar{F}$ is 1-1.

We claim that $\bar{F}\colon T^2 \to M$ is a diffeomorphism. Since $\bar{F}$ is $1-1$ it's enough to show that it's a local diffeomorphism. To see this it's enough to show that $d\bar{F}_p\colon T_pT^2 \to T_{F(p)}M$ is an injective for any $p \in T^2$. Indeed, Since $\dim T^2 = \dim M = 2$ this implies that $d\bar{F}_p$ is an isomorphism for any $p \in T^2$ and hence F is a local diffeomorphism by the Inverse Function theorem.

Let $i\colon M \to \mathbb{R}^3$ be the inclusion map. To see that $d\bar{F}_p\colon T_pT^2 \to T_{F(p)}M$ is injective it's enough to check that $di_{\bar{F}(p)} \circ d\bar{F}_p = d(i \circ \bar{F})_p$ is injective. Thus, it's enough to check that $d\bar{F}_p$ is injective when $\bar{F}$ is viewed as a map $T^2 \to \mathbb{R}^3$ and since the canonical projection $\pi\colon \mathbb{R}^2 \to T^2$ is a local diffeomorphism it's enough to check that $dF_{(\theta,\phi)}\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective for every $(\theta,\phi) \in \mathbb{R}^2$.

We compute

$$\frac{\partial F}{\partial \theta} = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta), \frac{\partial F}{\partial \phi} = (-(2+\cos\theta)\sin\phi, 2+\cos\theta)\cos\phi, 0)$$

Therefore,

$$\frac{\partial F}{\partial \theta} \times \frac{\partial F}{\partial \phi} = \det \begin{pmatrix} -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \\ -(2+\cos\theta)\sin\phi & (2+\cos\theta)\cos\phi & 0 \\ i & j & k \end{pmatrix} =$$

$$= ((2+\cos\theta)\cos\phi\cos\theta, -(2+\cos\theta)\sin\phi\cos\theta, -(2+\cos\theta)\sin\theta)$$

Therefore, $|\frac{\partial F}{\partial \theta} \times \frac{\partial F}{\partial \phi}|^2 = (2+\cos\theta)^2 \neq 0$ for any (θ,ϕ). Thus $\frac{\partial F}{\partial \theta}$, $\frac{\partial F}{\partial \phi}$ are linearly independent for any (θ,ϕ) i.e. $dF_{(\theta,\phi)}\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective for every $(\theta,\phi) \in \mathbb{R}^2$. $\square$.