Solutions to selected problems from homework 2

(1) Problem 2-5b from the book. Let $x\colon \mathbb{R} \to \mathbb{R}$ be given by $x(t) = t^3$. This is a homeomorphism and hence defines a smooth structure on $\mathbb{R}$ denoted by $\tilde{\mathbb{R}}$ given by a single chart x. Note that the parameterization $\phi = x^{-1}$ is given by $\phi(x) = x^{1/3}$.

Consider a smooth function $f\colon \mathbb{R} \to \mathbb{R}$. Prove that f is smooth with respect to the smooth structure $\tilde{\mathbb{R}}$ on the domain iff $f^{(n)}(0) = 0$ for any n not divisible by 3.

Solution

First, suppose f is smooth as a map $f\colon \tilde{\mathbb{R}} \to \mathbb{R}$. This means that $f \circ \phi\colon \mathbb{R} \to \mathbb{R}$ is smooth in ordinary sense. I.e. $g(x) = f(\sqrt[3]{x})$ is smooth on $\mathbb{R}$. Then $g(x^3) = f(x)$. Differentiating we get $f'(x) = g'(x^3) \cdot 3x^2, f''(x) = g''(x^3) \cdot 9x^4 + g'(x^3) \cdot 6x$. Plugging in $x = 0$ we get that $f'(0) = 0$ and $f''(0) = 0$. The case of a general derivative follows by induction by repeatedly differentiating the above formula.

Now suppose f is smooth in ordinary sense and satisfies $f^{(n)}(0) = 0$ for any n not divisible by 3.

We need to prove that $g(x) = f(\sqrt[3]{x})$ is smooth. Clearly g is smooth on $\mathbb{R}\backslash\{0\}$ as a composition of smooth functions. Thus, the only issue is to verify that g is smooth at 0.

Since $f'(0) = 0, f''(0) = 0$ we have that $f'(x)$ has first derivative 0 at 0 and therefore $f'(x)$ can be written $f'(x) = x^2 h(x)$ where h is smooth on $\mathbb{R}$. Moreover, Taylor series of f' at 0 is obtained from the Taylor series of h by multiplying by x^2. Therefore h satisfies the same condition on its derivatives as f, i.e. $h^{(n)}(0) = 0$ for any n not divisible by 3. Next, we can write $f(x) = f(0) + \int_0^x f'(t)dt$ and hence $g(x) = f(\sqrt[3]{x}) = f(0) + \int_0^{\sqrt[3]{x}} f'(t)dt$. Using the change of variables $t = \sqrt[3]{y}$ this gives $g(x) = f(0) + \int_0^{\sqrt[3]{x}} f'(t)dt = f(0) + \int_0^x f'(\sqrt[3]{y})\frac{1}{3\sqrt[3]{y^2}}dy$. Recalling that $f'(x) = x^2 h(x)$ this gives $g(x) = f(0) + \int_0^x f'(\sqrt[3]{y})\frac{1}{3\sqrt[3]{y^2}}dy = f(0) + \int_0^x \sqrt[3]{y^2} h(\sqrt[3]{y})\frac{1}{3\sqrt[3]{y^2}}dy = f(0) + \int_0^x h(\sqrt[3]{y})/3dy = f(0)+\int_0^x u(y)dy$ where $u(y) = h(\sqrt[3]{y})/3$. Since $u(y)$ is continuous this implies that g is C^1. However, the same argument applies to $u(y)$ because both f and h satisfy the same condition on their derivatives. Hence $u(y)$ is also C^1 which in turn implies that g is C^2. Repeatedly applying the same argument gives that g is C^k for any k.

(2) Let $U(n) = \{A \in M(n \times n, \mathbb{C})|$ such that $A \cdot A^* = Id\}$. Prove that $U(n)$ is a smooth manifold.

Hint. Consider the map $F\colon M(n\times n,\mathbb{C})\to M(n\times n,\mathbb{C})$ given by $F(A)=A\cdot A^*$. Then $U(n)=\{F=Id\}$. Notice that $F(A)$ is always self-adjoint.

Solution

Let $M(n\times n,\mathbb{C})$ be the space of all $n\times n$ matrices with complex coefficients. It's naturally isomorphic to $\mathbb{C}^{n^2}\cong\mathbb{R}^{2n^2}$. Let $S(n\times n,\mathbb{C})$ be the real vector space of all self-adjoint $n\times n$ complex matrices. Then it's easy to see that $S(n\times n,\mathbb{C})\cong\mathbb{R}^{n^2}$.

Consider the map $F\colon M(n\times n,\mathbb{C})\to M(n\times n,\mathbb{C})$ given by $F(A)=A\cdot A^*$. Note that $(A\cdot A^*)^*=A^{**}\cdot A^*=A\cdot A^*$. Thus $F(A)$ is always self-adjoint and we can view F as a map $F\colon \mathbb{R}^{2n^2}\cong M(n\times n,\mathbb{C})\to S(n\times n,\mathbb{C})\cong\mathbb{R}^{n^2}$. Then $U(n)=\{F=Id\}$. We claim that Id is a regular value of F.

Clearly F is smooth because it's given by polynomial equations in coordinates. Let us compute the differential of F at $A\in M(n\times n,\mathbb{C})$. Since F is smooth we have that for any $X\in M(n\times n,\mathbb{C}), A\in U(n)$ the value of $dF_A(X)$ is equal to the directional derivative $D_XF(A)=\lim_{t\to 0}\frac{F(A+tX)-F(A)}{t}=\lim_{t\to 0}\frac{(A+tX)(A^*+tX^*)-AA^*}{t}=\lim_{t\to 0}\frac{AA^*+t(AX^*+X^*A)+t^2XX^*-AA^*}{t}=AX^*+XA^*$.

we need to show that $df_A\colon M(n\times n,\mathbb{C})\to S(n\times n,\mathbb{C})$ is onto for any $A\in U(n)$. Given any $B\in S(n\times n,\mathbb{C})$ set $X=\frac{B(A^*)^{-1}}{2}$. Then $dF_A(X)=AX^*+XA^*=A[\frac{B(A^*)^{-1}}{2}]^*+\frac{B(A^*)^{-1}}{2}A^*=A[\frac{A^{-1}B^*}{2}]+\frac{B}{2}=B$ where in the last equality we used that $B=B^*$. Thus dF_A is onto for any $A\in U(n)$ and hence $U(n)$ is a manifold of dimension $2n^2-n^2=n^2$.

(3) Let $\pi\colon \mathbb{C}^{n+1}\backslash\{0\}\to\mathbb{CP}^n$ be the canonical projection map $\pi(z_0,\dots,z_n)=[z_0:\dots:z_n]$. Let M be a smooth manifold and let $f\colon\mathbb{CP}^n\to M$ be a map.

Prove that f is smooth if and only if $f\circ\pi\colon\mathbb{C}^{n+1}\backslash\{0\}\to M$ is smooth.

Solution

Since $\pi\colon\mathbb{C}^{n+1}\backslash\{0\}\to\mathbb{CP}^n$ is smooth it's obvious that if f is smooth then $f\circ\pi$ is also smooth as a composition of two smooth maps.

Now suppose $f\circ\pi$ is smooth and we need to show that f is smooth. Since smoothness is a local condition it's enough to show that $f|_{U_i}$ is smooth for every i where $U_i=\{[z_1:\dots:z_{n+1}]|z_i\neq 0\}, i=1,\dots,n+1$ is the standard atlas on $\mathbb{CP}^n$. We will only do it for $i=n+1$. the other is are treated in exactly the same way. We have a local smooth chart $x\colon U_{n+1}\to\mathbb{C}^n$ given by $x([z_1:\dots:z_{n+1}])=(z_1/z_{n+1},\dots,z_n/z_{n+1})$ with the inverse local parameterization $\phi=x^{-1}$ given by $\phi(u_1,\dots,u_n)=[u_1:\dots:u_n:1]$.

Consider the following ”section” map $s\colon U_{n+1} \to \mathbb{C}^{n+1}\setminus\{0\}$ given by $s(z_1 : \ldots : z_n : z_{n+1}) = (z_1/z_{n+1}, \ldots, z_n/z_{n+1}, 1)$. It's immediate to check that this map is well defined. Also $s \circ \phi = \phi$ which means that s is smooth. Lastly, $\pi \circ s = id|_{U_{n+1}}$. Therefore, $f|_{U_{n+1}} = f \circ (\pi \circ s) = (f \circ \pi) \circ s$ is smooth as a composition of two smooth maps.