Solutions to selected problems from homework 1

(1) Let $U \subset \mathbb{R}^{n+k}$ be open and let $F\colon U \to \mathbb{R}^k$ be a smooth map. Suppose $c \in \mathbb{R}^k$ is a regular value of F. It was proved in class that the level set $M = \{F = c\}$ admits a smooth atlas.

Prove that M is Hausdorff and admits a countable smooth atlas (and hence it is a smooth manifold).

Solution

By construction of the smooth structure on M, the topology on M is induced from $\mathbb{R}^{n+k}$, i.e. a subset $V \subset M$ is open in M iff there is an open $W \subset \mathbb{R}^{n+k}$ such that $V = M \cap W$. Since a subset of a Hausdorff space with induced topology is Hausdorff this means that M is Hausdorff.

Likewise, since $\mathbb{R}^{n+k}$ is second countable, any subset of $\mathbb{R}^{n+k}$ with induced topology is also second countable. In particular, M is second countable. Hence, if it admits a smooth atlas it also admits a countable smooth atlas.

(2) Let $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \max_i |x_i| = 1\}$ with induced topology from $\mathbb{R}^n$. Prove that X is a topological manifold of dimension $n-1$.

Solution

Consider the map $f\colon X \to \mathbb{S}^{n-1}$ given by $f(x) = \frac{x}{|x|}$. This map is obviously continuous. Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be given by $h(x) = ||x||_\infty = \max_i |x_i|$. It's easy to see that h is continuous.

It's also easy to check that $g\colon \mathbb{S}^n \to X$ given by $g(y) = \frac{x}{h(x)}$ is the inverse of f. Thus, $f\colon X \to \mathbb{S}^{n-1}$ is a homeomorphism. Since $\mathbb{S}^{n-1}$ is a topological manifold of dimension $n-1$, so is X.

(3) problem 1-7 from the book.

(a) The line through $N = (0, 0, \ldots, 1)$ and $x = (x^1, \ldots, x^{n+1})$ has the form $l(t) = N + t(x - N) = (0, 0, \ldots, 1) + t((x^1, \ldots, x^{n+1} - 1) = (tx^1, \ldots tx^n, 1 + t(x^{n+1} - 1))$. it intersects $x^{n+1} = 0$ when $1 + t(x^{n+1} - 1) = 0, t = \frac{1}{1-x^{n+1}}$ so that $l(t) = l(\frac{1}{1-x^{n+1}}) = (\frac{x^1}{1-x^{n+1}}, \ldots, \frac{x^1}{1-x^{n+1}}, 0)$ which gives that $\sigma(x) = (\frac{x^1}{1-x^{n+1}}, \ldots, \frac{x^1}{1-x^{n+1}})$.

(b) To see that the map σ is a bijection and to find its inverse, for a point $P = (u^1, \ldots, u^n, 0)$ consider the line $L(t)$ passing through P and N. It's given by $L(t) = N + t(P - N) = (tu^1, \ldots, tu^n, 1 - t)$. Let's find the intersection of L with the unit sphere. It occurs when $|L(t)|^2 = 1$, i.e. $1 = t^2((u^1)^2 + \ldots + (u^n)^2) + (1 - t)^2 = t^2((u^1)^2 + \ldots + (u^n)^2) + t^2 - 2t + 1$. This simplifies to $0 = t^2((u^1)^2 + \ldots + (u^n)^2) + t^2 - 2t$ which gives two solutions: $t = 0$ (this corresponds to $L(0) = N$) and $t = \frac{2}{1+(u^1)^2+\ldots+(u^n)^2} = \frac{2}{1+|u|^2}$. Therefore L intersects $\mathbb{S}^n \backslash \{N\}$ in precisely one point which means that σ is a bijection and $\sigma^{-1}(u) = L(\frac{2}{1+|u|^2}) =$

$(\frac{2u^1}{1+|u|^2}, \dots \frac{2u^n}{1+|u|^2}, \frac{|u|^2-1}{1+|u|^2})$. Since both maps are continuous they are homeomorphisms.

(c) We can similarly find the formula for the stereographic projection $\tilde{\sigma}$ from the south pole $S = (0, \dots, 0, -1)$. By composing with reflection in the hyperlane $x^{n+1} = 0$ it's obvious that $\tilde{\sigma}(x^1, \dots, x^n, x^{n+1}) = \sigma(x^1, \dots, x^n, -x^{n+1}) = \frac{1}{1+x^{n+1}}(x^1, \dots, x^n)$. It's straightforward to check that $\tilde{\sigma}(x) = -\sigma(-x)$. Using this we get that $\tilde{\sigma}^{-1}(u) = (\frac{2u^1}{1+|u|^2}, \dots \frac{2u^n}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2})$.
Thus $\tilde{\sigma}(\sigma^{-1}(u)) = \tilde{\sigma}(\frac{2u^1}{1+|u|^2}, \dots \frac{2u^n}{1+|u|^2}, \frac{1-|u|^2)}{1+|u|^2}) = \frac{1}{|u|^2}(u^1, \dots, u^n)$. This is a smooth map $\mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ with the inverse given by the same formula and hence $\sigma, \tilde{\sigma}$ give a smooth atlas on $\mathbb{S}^n$.