Term Test Solutions

(1) (12 pts) Mark True or False. You DO NOT need to justify your answers.
Let $M, N$ be smooth manifolds.
(a) Let $f: M \to N$ be smooth and let $c \in N$.
Then $S = \{ F = c \}$ is a smooth submanifold of $M$ if and only if $c$ is a regular value of $F$.

Solution
Counterexample: $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y^2$. 0 is not a regular value but $\{ f(x) = 0 \} = \{ y = 0 \}$ is a smooth submanifold of $\mathbb{R}^2$.
Answer: False.

(b) Let $M$ be compact. Then a smooth bijection $f: M \to N$ is a diffeomorphism.

Solution
Using partition of unity it’s easy to construct a smooth bijection $f: \mathbb{S}^1 \to \mathbb{S}^1$ which in local coordinates near $(1,0)$ is given by $f(t) = t^3$.
Answer: False.

(c) A smooth $1 − 1$ immersion $f: M \to N$ is a smooth embedding.

Solution
Counterexample: Figure eight map $f: (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin t, \sin 2t)$.
Answer: False.

(d) If $f: M \to N$ and $g: N \to P$ are submersions then $g \circ f: M \to P$ is a submersion.
Answer: True.

(e) If $P$ is a manifold with boundary then $\partial P$ is a manifold without boundary.
Answer: True.

(f) Let $A \subset M$ be a subset in $M$ and let $f: A \to \mathbb{R}$ be smooth. Then $f$ admits a smooth extension $\bar{f}: M \to \mathbb{R}$.

Solution
Counterexample: Let $\phi: (0, 2\pi) \to \mathbb{R}^2 = M$ be given by $\phi(t) = (\cos t, \sin t)$. This map is a smooth embedding. Let $A = \phi((0, 2\pi)) = \mathbb{S}^1 \setminus \{(1,0)\}$. Then the map $f: A \to \mathbb{R}$ given by $f(\phi(t)) = t$ is smooth but does not admit a continuous extension to $\mathbb{R}^2$.
Answer: False.
(2) (14 pts) Let \( M = \mathbb{R}^2 \) with standard coordinates \((x_1, x_2)\). Let \( p = (1, 1) \). Let 
\[
y = (y_1, y_2) : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be given by } y(x_1, x_2) = (x_1^2x_2, x_1 + x_2).
\]
(a) Show that there exists an open set \( U \subset M \) containing \( p \) such that \( V = y(U) \) is open in \( \mathbb{R}^2 \) and and \( y : U \to V \) is a diffeomorphism. Thus, \( y \) gives a coordinate chart on \( U \).
(b) Let \( v \in T_p M \) be given by 
\[
v = \frac{\partial}{\partial y_1} \big|_p.
\]

Find \( a, b \in \mathbb{R} \) such that 
\[
v = a \frac{\partial}{\partial x_1} \big|_p + b \frac{\partial}{\partial x_2} \big|_p.
\]

Solution
(a) We compute that the matrix of \( dy_p \) is given by 
\[
A = \begin{pmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}
\end{pmatrix}
\big|_{(1,1)} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]
Since \( \det A = 2 - 1 = 1 \neq 0 \), by the inverse function theorem \( f \) is a local diffeomorphism near \( p \) which proves part a). By part a) we have that 
\[
\begin{pmatrix}
\frac{\partial}{\partial y_1} \big|_p & \frac{\partial}{\partial y_2} \big|_p \\
\frac{\partial}{\partial x_1} \big|_p & \frac{\partial}{\partial x_2} \big|_p
\end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]

or
\[
\begin{cases}
\frac{\partial}{\partial x_1} \big|_p = 2 \frac{\partial}{\partial y_1} \big|_p + \frac{\partial}{\partial y_2} \big|_p \\
\frac{\partial}{\partial x_2} \big|_p = \frac{\partial}{\partial y_1} \big|_p + \frac{\partial}{\partial y_2} \big|_p
\end{cases}
\]
Solving this system for \( \frac{\partial}{\partial y_1} \big|_p \) we find
\[
\text{Answer: } \frac{\partial}{\partial y_1} \big|_p = \frac{\partial}{\partial x_1} \big|_p - \frac{\partial}{\partial x_2} \big|_p.
\]

(3) (12 pts) Let \( M \) be a smooth manifold and let \( A \subset M \) be a closed subset. Let 
\( f : A \to \mathbb{R} \) be a smooth function such that \( f(x) > 0 \) for any \( x \in A \).

Prove that there exists a smooth function \( \bar{f} : M \to \mathbb{R} \) such that \( \bar{f}|_A = f \) and 
\( \bar{f}(x) > 0 \) for any \( x \in M \).

Solution 1
Let \( h = \ln f \). Then \( h \) is clearly smooth on \( A \) as a composition of smooth functions.

By a theorem from class there exists a smooth function \( \bar{h} : M \to \mathbb{R} \) such that \( \bar{h}|_A = h \). Then \( \bar{f} := e^\bar{h} \) is a smooth positive extension of \( f \).

Solution 2
Let \( U_0 = M \setminus A \). Then \( U_0 \) is open. Set \( f_0 : U_0 \to \mathbb{R} \) be \( f_0(x) \equiv 1 \).

By definition of a smooth function on \( A \), for any \( p \in A \) there is an open set \( U_p \) containing \( p \) and a smooth map \( f_p : U \to \mathbb{R} \) such that \( f_p|_{U_p \cap A} = f|_{U_p \cap A} \). By possibly making \( U_p \) smaller we can assume that \( \bar{f}_p \) is positive on \( U_p \). Let \( \{\phi_i\}_{i=1}^\infty \) be a partition of unity subordinate to the cover of \( M \) given by \( U_0 \) and \( \{U_p\}_{p \in A} \).
Then for every $i$ we have that $\text{supp } \phi_i \subset U_0$ or $\text{supp } \phi_i \subset U_{p_i}$ for some $p_i \in A$.
Set $f_i = f_0$ in the former case and $f_i = f_{p_i}$ in the latter case.
Then $\tilde{f} = \sum_i \phi_i f_i$ is a smooth positive extension of $f$.

(4) (12 pts) Let $S = \{ (x, y) \in \mathbb{R}^2 : |x|^3 + |y|^3 = 1 \}$. Is $S$ a smooth submanifold of $\mathbb{R}^2$? Justify your answer.

Solution

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = |x|^3 + |y|^3$. Let $U = \{ (x, y) : x \neq 0, y \neq 0 \}$. Then $U$ is open in $\mathbb{R}^2$ and $f_U$ is smooth. Also, it’s easy to see that 1 is a regular value of $f|_U$. Thus $S \cap U$ is a smooth submanifold of dimension 1 in $U$. Therefore, if $S$ were a smooth submanifold of $\mathbb{R}^2$ it could only be a submanifold of dimension 1. Suppose that’s the case. Let $p = (0, 1)$. By a theorem from class in a small open neighborhood of $p$ $S$ must be either be a graph of a smooth function $y = y(x)$ or a smooth function $x = x(y)$. The latter alternative is false since $S$ is not a graph of a function $x = x(y)$ near $p$ because if $(x, y) \in S$ then $(-x, y) \in S$ also.

The former alternative is false since near $p$ it holds that $S$ is given by the graph of $y = \sqrt[3]{1 - |x|^3}$ which is not smooth in $x$. Indeed, let $h(x) = \sqrt[3]{1 - |x|^3}$. Then

$$h(x) = \begin{cases} \sqrt[3]{1 - x^3} & \text{if } x \geq 0 \\ \sqrt[3]{1 + x^3} & \text{if } x \leq 0 \end{cases}$$

Note that the Taylor series at 0 for $\sqrt[3]{1 - x}$ is $1 - \frac{1}{3} x + \ldots$ and hence he Taylor series at 0 for $\sqrt[3]{1 - x^3}$ is $1 - \frac{1}{3} x^3 + \ldots$.

Similarly, the Taylor series at 0 for $\sqrt[3]{1 + x}$ is $1 + \frac{1}{3} x + \ldots$ and hence the one for $\sqrt[3]{1 + x^3}$ is $1 + \frac{1}{3} x^3 + \ldots$.

Since these do not agree, $h(x)$ is not smooth at 0.

Answer: $S$ is not a smooth submanifold of $\mathbb{R}^2$. 