Term Test Solutions

(1) (12 pts) Mark **True or False**. You **DO NOT** need to justify your answers.
Let M, N be smooth manifolds.

(a) Let $f\colon M \to N$ be smooth and let $c \in N$.
Then $S = \{F = c\}$ is a smooth submanifold of M if and only if c is a regular value of F.

Solution

Counterexample: $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y^2$. 0 is not a regular value but $\{f(x) = 0\} = \{y = 0\}$ is a smooth submanifold of $\mathbb{R}^2$.
Answer: False.

(b) Let M be compact. Then a smooth bijection $f\colon M \to N$ is a diffeomorphism.

Solution

Using partition of unity it's easy to construct a smooth bijection $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ which in local coordinates near $(1, 0)$ is given by $f(t) = t^3$.
Answer: False.

(c) A smooth $1-1$ immersion $f\colon M \to N$ is a smooth embedding.

Solution

Counterexample: Figure eight map $f\colon (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin t, \sin 2t)$.
Answer: False.

(d) If $f\colon M \to N$ and $g\colon N \to P$ are submersions then $g \circ f\colon M \to P$ is a submersion.
Answer: True.

(e) If P is a manifold with boundary then ∂P is a manifold without boundary.
Answer: True.

(f) Let $A \subset M$ be a subset in M and let $f\colon A \to \mathbb{R}$ be smooth. Then f admits a smooth extension $\bar{f}\colon M \to \mathbb{R}$.

Solution

Counterexample: Let $\phi\colon (0, 2\pi) \to \mathbb{R}^2 = M$ be given by $\phi(t) = (\cos t, \sin t)$. This map is a smooth embedding. Let $A = \phi((0, 2\pi)) = \mathbb{S}^1 \backslash \{(1, 0)\}$. Then the map $f\colon A \to \mathbb{R}$ given by $f(\phi(t)) = t$ is smooth but does not admit a continuous extension to $\mathbb{R}^2$.
Answer: False.

(2) (14 pts) Let $M = \mathbb{R}^2$ with standard coordinates (x_1, x_2). Let $p = (1,1)$. Let $y = (y_1, y_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $y(x_1, x_2) = (x_1^2 x_2, x_1 + x_2)$.

(a) Show that there exists an open set $U \subset M$ containing p such that $V = y(U)$ is open in $\mathbb{R}^2$ and and $y\colon U \to V$ is a diffeomorphism. Thus, y gives a coordinate chart on U.

(b) Let $v \in T_pM$ be given by $v = \frac{\partial}{\partial y_1}|_p$.
Find $a, b \in \mathbb{R}$ such that $v = a\frac{\partial}{\partial x_1}|_p + b\frac{\partial}{\partial x_2}|_p$.

Solution

(a) We compute that the matrix of dy_p is given by

$$A = [\frac{\partial y_i}{\partial x_i}|_p] = \begin{pmatrix} 2x_1x_2 & x_1^2 \\ 1 & 1 \end{pmatrix} |_{(1,1)} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Since $\det A = 2 - 1 = 1 \neq 0$, by the inverse function theorem f is a local diffeomorphism near p which proves part a). By part a) we have that

$$\begin{pmatrix} \frac{\partial}{\partial y_1}|_p & \frac{\partial}{\partial y_2}|_p \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1}|_p & \frac{\partial}{\partial x_2}|_p \end{pmatrix}$$

or

$$\begin{cases} \frac{\partial}{\partial x_1}|_p = 2\frac{\partial}{\partial y_1}|_p + \frac{\partial}{\partial y_2}|_p \\ \frac{\partial}{\partial x_2}|_p = \frac{\partial}{\partial y_1}|_p + \frac{\partial}{\partial y_2}|_p \end{cases}$$

Solving this system for $\frac{\partial}{\partial y_1}|_p$ we find
Answer: $\frac{\partial}{\partial y_1}|_p = \frac{\partial}{\partial x_1}|_p - \frac{\partial}{\partial x_2}|_p$.

(3) (12 pts) Let M be a smooth manifold and let $A \subset M$ be a closed subset. Let $f\colon A \to \mathbb{R}$ be a smooth function such that $f(x) > 0$ for any $x \in A$.

Prove that there exists a smooth function $\bar{f}\colon M \to \mathbb{R}$ such that $\bar{f}|_A = f$ and $\bar{f}(x) > 0$ for any $x \in M$.

Solution 1

Let $h = \ln f$. Then h is clearly smooth on A as a composition of smooth functions. By a theorem from class there exists a smooth function $\bar{h}\colon M \to \mathbb{R}$ such that $\bar{h}|_A = h$. Then $\bar{f} := e^{\bar{h}}$ is a smooth positive extension of f.

Solution 2

Let $U_0 = M \backslash A$. Then U_0 is open. Set $f_0\colon U_0 \to \mathbb{R}$ be $f_0(x) \equiv 1$.

By definition of a smooth function on A, for any $p \in A$ there is an open set U_p containing p and a smooth map $f_p\colon U \to \mathbb{R}$ such that $f_p|_{U_p \cap A} = f|_{U_p \cap A}$. By possibly making U_p smaller we can assume that $\bar{f}_p$ is positive on U_p. Let $\{\phi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to the cover of M given by U_0 and $\{U_p\}_{p \in A}$.

Then for every i we have that supp $\phi_i \subset U_0$ or supp $\phi_i \subset U_{p_i}$ for some $p_i \in A$. Set $f_i = f_0$ in the former case and $f_i = f_{p_i}$ in the latter case.

Then $\bar{f} = \Sigma_i \phi_i f_i$ is a smooth positive extension of f.

(4) (12 pts) Let $S = \{(x,y) \in \mathbb{R}^2 : |x|^3 + |y|^3 = 1\}$. Is S a smooth submanifold of $\mathbb{R}^2$? Justify your answer.

Solution

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = |x|^3 + |y|^3$. Let $U = \{(x,y) : x \neq 0, y \neq 0\}$. Then U is open in $\mathbb{R}^2$ and f_U is smooth. also, it's easy to see that 1 is a regular value of $f|_U$. Thus $S \cap U$ is a smooth submanifold of dimension 1 in U. Therefore, if S were a smooth submanifold of $\mathbb{R}^2$ it could only be a submanifold of dimension 1. Suppose that's the case. let $p = (0,1)$. By a theorem from class in a small open neighborhood of p S must be either be a graph of a smooth function $y = y(x)$ or a smooth function $x = x(y)$. The latter alternative is false since S is not a graph of a function $x = x(y)$ near p because if $(x,y) \in S$ then $(-x,y) \in S$ also.

The former alternative is false since near p it holds that S is given by the graph of $y = \sqrt[3]{1 - |x|^3}$ which is not smooth in x. Indeed, let $h(x) = \sqrt[3]{1 - |x|^3}$. Then

$$h(x) = \begin{cases} \sqrt[3]{1 - x^3} \text{ if } x \geq 0 \\ \sqrt[3]{1 + x^3} \text{ if } x \leq 0 \end{cases}$$

Note that the Taylor series at 0 for $\sqrt[3]{1 - x}$ is $1 - \frac{1}{3}x + \ldots$ and hence he Taylor series at 0 for $\sqrt[3]{1 - x^3}$ is $1 - \frac{1}{3}x^3 + \ldots$.

Similarly, the Taylor series at 0 for $\sqrt[3]{1 + x}$ is $1 + \frac{1}{3}x + \ldots$ and hence the one for $\sqrt[3]{1 + x^3}$ is $1 + \frac{1}{3}x^3 + \ldots$.

Since these do not agree, $h(x)$ is not smooth at 0.

Answer: S is not a smooth submanifold of $\mathbb{R}^2$.