Solution to problem 14-2

Let V be an n-dimensional vector space. Claim: For $k = 0, 1, n-1, n$ every element of $A^k(V)$ is decomposable and for $k \neq 0, 1, n-1, n$ not every element of $A^k(V)$ is decomposable.

The hard part is to prove the second part of the above statement.

Let $e_1, \dots e_n$ be a basis of V and let $e^1, \dots e^n$ be the dual basis of $V^\star$.

We'll need the following general observation

Lemma 0.1. *Let $w_1, \dots w_k \in V^\star$ be linearly independent. Let $W = \operatorname{span}\{w_1, \dots, w_k\}$. Let $w'_1, \dots w'_k$ be another basis of W then $w'_1 \wedge \dots \wedge w'_k = \det A \cdot w_1 \wedge \dots \wedge w_k$ where A is the transition matrix from $w_1, \dots w_k$ to $w'_1, \dots w'_k$*

The proof is easy is left to the reader. Let's prove that not every vector in $A^k(V)$ is decomposable if $2 \leq k \leq n-2$. We will give two different proofs.

Proof 1

First, let's consider the case $n = 4$. We need to show that not every element of $A^2(V)$ is decomposable. **Claim:** $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4$ is not decomposable.

Suppose $\alpha = w_1 \wedge w_2$ for some $w_1, w_2 \in V^\star$. Let $W = \operatorname{span}\{w_1, w_2\}$.

Let $V_{13} = \operatorname{span}\{e_1, e_3\}$ and let $i\colon V_{13} \to V$ be the canonical inclusion. Look at $i_1^*\colon A^1(V) \to A^1(V_{13})$. It's easy to see that this map is onto and its kernel is equal to $\operatorname{span}\{e^2, e^4\}$. Hence we have that $i_2^*(\alpha) = i^*(e^1 \wedge e^2 + e^3 \wedge e^4) = 0$. On the other hand, since $\alpha = w_1 \wedge w_2$ we have that $0 = i^*(\alpha) = i^*(w_1 \wedge w_2) = i^*(w_1) \wedge i^*(w_2)$. Therefore, $i^*(w_1), i^*(w_2)$ are linearly dependent and hence $\ker i_1^* \cap W \neq 0$. In other words there is a nonzero convector in W of the form $\lambda_1 e^2 + \lambda_2 e^4$.

Similarly looking at the inclusion of $\operatorname{span}\{e_1, e_4\}$ into V we can find a nonzero element in W of the form $\mu_1 e^2 + \mu_2 e^3$. Likewise W contains nonzero vectors of the form $\nu_1 e^1 + \nu_2 e^3$ and $\eta_1 e^1 + \eta_2 e^4$.

Case 1.

Suppose one of the numbers $\lambda_i, \mu_i, \nu_i, \eta_i$ is equal to 0.

Then W contains one of the basic convectors e^1, e^2, e^3 or e^4. Say, it contains e^1. We can find a new basis w'_1, w'_2 of W with $w'_1 = e_1$ and such that the transition matrix from w_1, w_2 to w'_1, w'_2 has determinant 1. Then $e^1 \wedge e^2 + e^3 \wedge e^4 = w_1 \wedge w_2 = w'_1 \wedge w'_2 = e^1 \wedge w'_2$. We can write w'_2 in the standard basis of $V^\star$ as $w_2 = ae^1 + be^2 + ce^3 + de^4$. Then $e^1 \wedge w'_2 = be^1 \wedge e^2 + ce^1 \wedge e^3 + de^1 \wedge e^4 \neq e^1 \wedge e^2 + e^3 \wedge e^4$ since $\{e^i \wedge e^j\}_{\{i<j\}}$ is a basis of $A^2(V)$.

Case 2.

Now suppose that none of $\lambda_i, \mu_i, \nu_i, \eta_i$ are 0. It's easy to see that this is impossible since $\dim W = 2$ but we will give a different argument which generalizes to $n \geq 5$ and $2 \leq k \leq n-2$. By rescaling we can assume that $\lambda_2 = \mu_2 = \nu_2 = \eta_2 = 1$. Then W contains vectors of the form $e^3 + \nu e^1$ and $e^3 + \mu e^2$ and hence it also contains $\nu e^1 - \mu e^2$ and $e^2 - ae^1$ where $a = \nu/\mu$.

Let $v_1, \dots v_4$ be another basis of V given by $v_1 = e_1 + ae_2, v_i = e_i$ for $i = 2, 3, 4$. It's straightforward to check that the dual basis is $v^1 = e^1, v^2 = e^2 - ae^1, v^3 = e^3, v^4 = e^4$.

Then we still have $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4 = e^1 \wedge (e^2 - ae^1) + e^3 \wedge e^4 = v^1 \wedge v^2 + v^3 \wedge v^4$ but now we have that $v_2 \in W$. This reduces the problem to Case 1 which is already handled.

The above proof easily generalizes to show that for general n, k if $I = (i_1, \dots, i_{k-2})$ is a multi-index not containing $1, 2, 3, 4$ then $\alpha = (e^1 \wedge e^2 + e^3 \wedge e^4) \wedge e^I$ is indecomposable. It's clearly enough to prove it for $k = n - 2$ and $I = (5, 6, \dots n)$.

Suppose $\alpha = w_1 \wedge \dots \wedge w_{n-2}$ for some linearly independent $w_1, \dots w_{n-2} \in V$. Again, let $V_{13} = \operatorname{span}\{e_1, e_3, e_5, \dots e_n\}$ and let $i\colon V_{13} \to V$ be the inclusion. Again looking at $i_1^*\colon A^1(V) \to A^1(V_{13})$ we see that this map is onto and its kernel is equal to $\operatorname{span}\{e^2, e^4\}$. Again $i_2^*(\alpha) = 0$ and hence $\ker i_1^* \cap W \neq 0$. Therefore W contains a nonzero element of the form $\lambda_1 e^2 + \lambda_2 e^4$. The rest of the proof is the same as in the case $n = 4, k = 2$. $\square$

Proof 2

Let $A^i = A^i(V)$ be the space of alternating i-tensors on V. Consider the map $F\colon A^1 \times A^1 \times \dots \times A^1 \to A^k$ given by $F(w_1, w_2, \dots, w_k) = w_1 \wedge \dots \wedge w_k$. We want to show that if $2 \leq k \leq n-2$ then F is not onto. When written in coordinates F is easily seen to have polynomial components and hence it is smooth. Let $U \subset A^1 \times A^1 \times \dots \times A^1 \cong \mathbb{R}^{nk}$ be the set of linearly independent k-tuples $(w_1, w_2, \dots, w_k)$.

Then $U = \{F \neq 0\}$ is open in $\mathbb{R}^{nk}$ and $F\colon \mathbb{R}^{nk} \to A^k \backslash \{0\} \cong \mathbb{R}^{\binom{n}{k}} \backslash \{0\}$. By Lemma 0.1, if $w_1', \dots w_k'$ be another basis of $W = \operatorname{span}\{w_1, \dots, w_k\}$ then $F(w_1', \dots, w_k') = \lambda F(w_1, \dots, w_k)$ for a nonzero λ. Therefore F induces a well-defined map $\bar{F}\colon Gr_k(V^*) \to \mathbb{RP}^{\binom{n}{k}-1}$ where $Gr_k(V^*)$ is the Grassmanian of k-planes in V^*. By Example 1.36 from the book $Gr_k(V^*)$ is a smooth manifold of dimension $k(n-k)$. Moreover, by looking at $\bar{F}$ in the local coordinates described in Example 1.36 it's easy to see that $\bar{F}$ is smooth.

It's enough to show that $\bar{F}$ is not onto. This is true because for $2 \leq k \leq n-2$ we have $k(n-k) < \binom{n}{k} - 1$ and hence $\bar{F}$ is not onto by Sard's Theorem. $\square$.

Lastly, let us show that every element in $A^{n-1}(V)$ is decomposable. Note that any element in $A^{n-1}(V)$ can be written as $\sum_I \lambda_I e^I$ and each $\lambda_I e^I$ is decomposable. Therefore it's enough to show that if $\alpha, \beta \in A^{n-1}(V)$ are decomposable then $\alpha + \beta$ is also decomposable. The general case then follows by induction on the number of summands.

Suppose $\alpha = w_1 \wedge w_2 \wedge \dots \wedge w_{n-1} \neq 0$ and $\beta = w_1' \wedge w_2' \wedge \dots \wedge w_{n-1}' \neq 0$. Let $W = \operatorname{span}\{w_1, \dots, w_{n-1}\}, W' = \operatorname{span}\{w_1', \dots, w_{n-1}'\}$. Let $U = W \cap W'$. Since $\dim V = n, \dim W = \dim W' = n - 1$ we have that $\dim U \geq n - 2$. If $\dim U = n - 1$ then $W = W'$ and α, β are proportional by Lemma 0.1. Therefore $\alpha + \beta$ is also a multiple of α and hence is decomposable.

Suppose $\dim U = n - 2$.

Let $u_1, \ldots, u_{n-2}$ be a basis of U. We can find $w \in W$ such that $u_1, \ldots, u_{n-2}, w$ is a basis of W. By lemma 0.1 by rescaling w by a constant we can assume that $w_1 \wedge w_2 \wedge \ldots \wedge w_{n-1} = u_1 \wedge u_2 \wedge \ldots \wedge u_{n-1} \wedge w$.

Similarly, we can find $w' \in W'$ such that $w'_1 \wedge w_2 \wedge \ldots \wedge w'_{n-1} = u_1 \wedge u_2 \wedge \ldots \wedge u_{n-1} \wedge w'$. Therefore, $\alpha + \beta = u_1 \wedge u_2 \wedge \ldots \wedge u_{n-1} \wedge w + u_1 \wedge u_2 \wedge \ldots \wedge u_{n-1} \wedge w' = \alpha + \beta = u_1 \wedge u_2 \wedge \ldots \wedge u_{n-1} \wedge (w + w')$ is decomposable. $\square$.