INTRODUCTION TO TRANSCENDENTAL NUMBERS

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Abstract. The study of transcendental numbers has developed into an enriching theory and constitutes an important part of mathematics. This report aims to give a quick overview about the theory of transcendental numbers and some of its recent developments. The main focus is on the proof that e is transcendental. The Hilbert's seventh problem will also be introduced.

1. Introduction

Transcendental number theory is a branch of number theory that concerns about the transcendence and algebraicity of numbers. Dated back to the time of Euler or even earlier, it has developed into an enriching theory with many applications in mathematics, especially in the area of Diophantine equations.

Whether there is any transcendental number is not an easy question to answer. The discovery of the first transcendental number by Liouville in 1851 sparked up an interest in the field and began a new era in the theory of transcendental number. In 1873, Charles Hermite succeeded in proving that e is transcendental. And within a decade, Lindemann established the transcendence of π in 1882, which led to the impossibility of the ancient Greek problem of squaring the circle. The theory has progressed significantly in recent years, with answer to the Hilbert's seventh problem and the discovery of a nontrivial lower bound for linear forms of logarithms of algebraic numbers.

Although in 1874, the work of Georg Cantor demonstrated the ubiquity of transcendental numbers (which is quite surprising), finding one or proving existing numbers are transcendental may be extremely hard. In this report, we will focus on the proof that e is transcendental.

2. Definitions

Definition 2.1 (Algebraic numbers)**.** A complex number α is called an *algebraic number* of degree n if it is a root of a polynomial

$$a_0 + a_1 x + \cdots + a_n x^n = 0,$$

where $a_0, a_1, \cdots, a_n,\ a_n \neq 0$ are integers (or equivalently, rational numbers), and

$$a_0 + a_1 \alpha + \cdots + a_m \alpha^m \neq 0,$$

for any integers $a_0, a_1, \cdots, a_m,\ a_m \neq 0$ and $m < n$.

Definition 2.2 (Transcendental numbers)**.** A number which is not an algebraic number of any degree is called a *transcendental number*. In other words, a number is transcendental if it is not a root of any nonzero polynomial with integer coefficients.

Example.

- A rational number p/q is algebraic because it's the root of the equation $qx - p = 0$.
- The golden ratio $\alpha = \dfrac{1+\sqrt{5}}{2}$ is an algebraic number of degree 2 because it's the root of the equation $x^2 - x - 1 = 0$.

3. The First Transcendental Number

Euler was probably the first person to define transcendental numbers in the modern sense [4]. However, the existence of transcendental numbers was not confirmed until 1851 when Joseph Liouville, a French mathematician, gave the first example of transcendental numbers, the Liouville constant:

$$\alpha = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots,$$

where the 1's appear at the $1st, 2nd, \cdots, k!th, \cdots$ positions after the decimal point. We can verify that this number is transcendental using the Liouville's approximation theorem.

Theorem 3.1 (Liouville's approximation theorem)**.** *For any real algebraic number α with degree $n > 1$, there exists $c = c(\alpha) > 0$ such that $|\alpha - p/q| > c/q^n$ for all integers p and q $(q > 0)$.*

The proof for the above theorem will not be presented here. Interested readers can refer to [1] for a complete proof. To show that α is transcendental, we first observe that α is irrational. Suppose it is algebraic of degree n, then by *Theorem 3.1*, there exists $c > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n} \quad \forall \frac{p}{q} \in \mathbb{Q},\ q > 0. \tag{1}$$

Let $j = n + i$ for some positive integer i and i is large enough such that $10^{-j!i} < c$. Now we choose

$$p = 10^{j!} \sum_{k=1}^{j} 10^{-k!}, \quad q = 10^{j!}.$$

Then

$$\begin{aligned} \left|\alpha - \frac{p}{q}\right| &= \sum_{k=j+1}^{\infty} 10^{-k!} \\ &< 10^{-(j+1)!}\left(\frac{1}{10} + \frac{1}{10^2} + \cdots\right) \\ &< 10^{-j!j} = 10^{-j!n} 10^{-j!i} \\ &< c10^{-j!n} = \frac{c}{q^n}. \end{aligned}$$

We get a contradiction and thus α is a transcendental number. □

The Liouville constant is an instance of a special class of numbers, the *Liouville numbers* [3]. These are all the real numbers x with the property that for any positive integer n, there exists a rational number p/q $(q > 1)$ such that $0 < |x - p/q| < 1/q^n$. Using similar technique, we can show that indeed all Liouville numbers are transcendental.

Liouville theorem tells us that there is a bound for the approximation of irrational algebraic number by rational numbers. In other words, the better the approximation by rational numbers, the more likely the number is transcendental. This idea can be carried on to establishing the transcendence of e in the next section.

4. The Transcendence of e

Theorem 4.1. *e is transcendental.*

The number e, sometimes called Euler's number or Napier's constant, appears quite frequently in mathematics. Although the irrationality of e was early shown by Euler in 1744, it was not until 1873 that the transcendence of e was established by Charles Hermite. The work of Hermite was later simplified by David Hilbert in 1893. The proof that we are going to present here is derived from [2]. The idea is to assume that e is an algebraic number of degree n, i.e. it satisfies the equation

$$a_0 + a_1 e + \cdots + a_n e^n = 0,$$

where all the a_i's are integers and $a_n \neq 0$. We will approximate $e, e^2, \cdots, e^n$ by finding suitable integers $M, M_1, M_2, \cdots, M_n$ and certain small positive numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ such that

$$e^k = \frac{M_k + \epsilon_k}{M}, \quad \text{for } k = 1, 2, \cdots, n$$

and arrive at the contradiction. To facilitate the proof, we first establish several results.

Lemma 4.1. *Let n be a positive integer and $p > n$ be a prime number. Show that*

$$M = \frac{1}{(p-1)!} \int_0^\infty x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p e^{-x}\, dx$$

is an integer not divisible by p.

Proof. In order to prove the lemma, we need one property of the *Gamma function*

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t} dt = n! \quad \text{for nonnegative integers } n. \tag{2}$$

Let $P(x)$ denote the polynomial in the integrand, we have

$$\begin{aligned} P(x) &= x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p \\ &= x^{p-1}(x^n + \cdots \pm n!)^p \\ &= x^{p-1}(x^{np} + \cdots \pm (n!)^p) \\ &= x^{p-1+np} + \cdots \pm (n!)^p x^{p-1}. \end{aligned}$$

Thus $P(x)$ can be written in the form

$$P(x) = \sum_{j=0}^{np} m_j x^{p-1+j},$$

where all the m_j's are integers, and $m_0 = \pm(n!)^p$. Substituting $P(x)$ into M and apply (2), we get

$$\begin{aligned} M &= \frac{1}{(p-1)!} \int_0^\infty \sum_{j=0}^{np} m_j x^{p-1+j} e^{-x}\, dx \\ &= \frac{1}{(p-1)!} \sum_{j=0}^{np} \int_0^\infty m_j x^{p-1+j} e^{-x}\, dx \\ &= \sum_{j=0}^{np} \frac{m_j (p-1+j)!}{(p-1)!} \\ &= m_0 + \sum_{j=1}^{np} \frac{m_j (p-1+j)!}{(p-1)!}. \end{aligned}$$

We see that starting from $j = 1$, every term of M is divisible by p (each term contains p as a factor). However, the first term $m_0 = \pm(n!)^p$ is not divisible by p since $p > n$. Thus M is not divisible by p. $\square$

Lemma 4.2. *Let n be a positive interger and p be a prime number. Show that for $k = 1, 2, \cdots, n$,*

$$M_k = \frac{e^k}{(p-1)!} \int_k^\infty x^{p-1} [(x-1)(x-2)\cdots(x-n)]^p e^{-x}\, dx$$

is an integer divisible by p.

Proof. Let $t = x - k$, we have

$$\begin{aligned} M_k &= \frac{e^k}{(p-1)!} \int_0^\infty (t+k)^{p-1} [(t+k-1)(t+k-2)\cdots(t+k-n)]^p e^{-(t+k)}\, dt \\ &= \frac{1}{(p-1)!} \int_0^\infty (t+k)^{p-1} [(t+k-1)(t+k-2)\cdots(t+k-n)]^p e^{-t}\, dt. \end{aligned}$$

The polynomial in the integrand contains the factor t^p (since $k = 1, 2, \cdots, n$). Thus it has the degree at most $np + p - 1$ and at least p. We can rewrite it in the form

$$\sum_{j=1}^{np} m_j t^{p-1+j},$$

where all the m_j's are integers. We have

$$\begin{aligned} M_k &= \frac{1}{(p-1)!}\int_0^\infty \sum_{j=1}^{np} m_j t^{p-1+j} e^{-t}\, dt \\ &= \frac{1}{(p-1)!}\sum_{j=1}^{np}\int_0^\infty m_j t^{p-1+j} e^{-t}\, dt \\ &= \sum_{j=1}^{np} \frac{m_j (p-1+j)!}{(p-1)!}. \end{aligned}$$

As each term of M_k is an integer containing p as a factor, M_k is divisible by p. □

Lemma 4.3. *Let n be a positive integer and p be a prime number. Show that for $k = 1, 2, \cdots, n$,*

$$|\epsilon_k| = \left|\frac{e^k}{(p-1)!}\int_0^k x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p e^{-x}\, dx\right|$$

can be made as small as possible by choosing p large enough.

Proof. We have

$$\begin{aligned} |\epsilon_k| &= \left|\frac{e^k}{(p-1)!}\int_0^k x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p e^{-x}\, dx\right| \\ &\le \frac{e^k}{(p-1)!}\int_0^k \left|x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p e^{-x}\right|\, dx \\ &\le \frac{e^n}{(p-1)!}\int_0^k A e^{-x}\, dx, \quad A = \max_{0\le x\le n}\left|x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p\right| \\ &\le \frac{e^n A}{(p-1)!} \quad \text{since} \int_0^k e^{-x}\, dx = 1 - e^{-k} < 1. \end{aligned}$$

Because $e^n A$ is just a constant, we can make $e^n A/(p-1)!$ as small as we like by choosing p large enough. This can be done since there are infinitely many prime numbers. □

With the above 3 lemmas, we are now ready to prove that e is transcendental. Assume that e is an algebraic number of degree n, i.e. there exists integers $a_0, a_1, \cdots, a_n$ with $a_n \ne 0$ such that

$$a_0 + a_1 e + \cdots + a_n e^n = 0. \tag{3}$$

Let M, M_k, ϵ_k be defined as in *Lemma 4.1*, *Lemma 4.2*, *Lemma 4.3* respectively. It can be verified that for $k = 1, 2, \cdots, n$,

$$e^k = \frac{M_k + \epsilon_k}{M}. \tag{4}$$

Substituting (4) into (3) we get

$$\begin{aligned} & a_0 + a_1\left(\frac{M_1+\epsilon_1}{M}\right) + a_2\left(\frac{M_2+\epsilon_2}{M}\right) + \cdots + a_n\left(\frac{M_n+\epsilon_n}{M}\right) = 0 \\ \Rightarrow\ & (a_0 M + a_1 M_1 + \cdots + a_n M_n) + (a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n) = 0. \end{aligned}$$

We know from the lemmas that M is not divisible by p and if we choose $p > |a_0|$, then a_0M is not divisible by p. However, all the M_k's are divisible by p. Thus the first term in brackets of the LHS of the above expression is not divisible by p and therefore is a nonzero integer. On the other hand, the second term in brackets of the LHS can be made arbitrarily small, since each ϵ_k can be made arbitrarily small (all we need to do is to choose some suitable prime number p). In particular, we can make the magnitude of the second term $< 1/2$ and hence their sum cannot be 0, which is a contradiction. □

5. Further Developments

One year after Charles Hermite established the transcendence of e, Georg Cantor showed that the set of all real numbers is uncountable, which led to the fact that almost every number is transcendental (as the set of all real algebraic numbers is countable). In 1882, Lindemann succeeded in generalizing Hermite's method to prove that π is transcendental, which resolved the ancient Greek problem of squaring the circle. In 1900, David Hilbert posed 23 influential problems. One of which, problem 7, concerns about the transcendence of numbers of the form α^β, where α and β are algebraic numbers with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$. The answer for this problem was provided in 1934 by the Gelfond–Schneider theorem:

Theorem 5.1 (Gelfond–Schneider theorem)**.** *If α and β are algebraic numbers, where $\alpha \neq 0, 1$ and β is not rational, then α^β is transcendental.*

The theorem immediately shows the transcendence of $2^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}}$ and e^π (recall that $e^\pi = (i)^{-2i}$). The theory of transcendental numbers has made major advance in recent years. Among many great contributors like Mordell, Siegel, Mahler, Gelfond, Schneider, Schmidt, etc. is Alan Baker, who found an effective lower bound for the linear form of logarithms of algebraic numbers in 1966, which allows for the generalization of the Gelfond–Schneider theorem and contributes significantly to the area of Diophantine equations. The complete treatment of this result can be found in [1]. Finally, the list of 15 most famous transcendental numbers can be found at

http://sprott.physics.wisc.edu/Pickover/trans.html

References

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