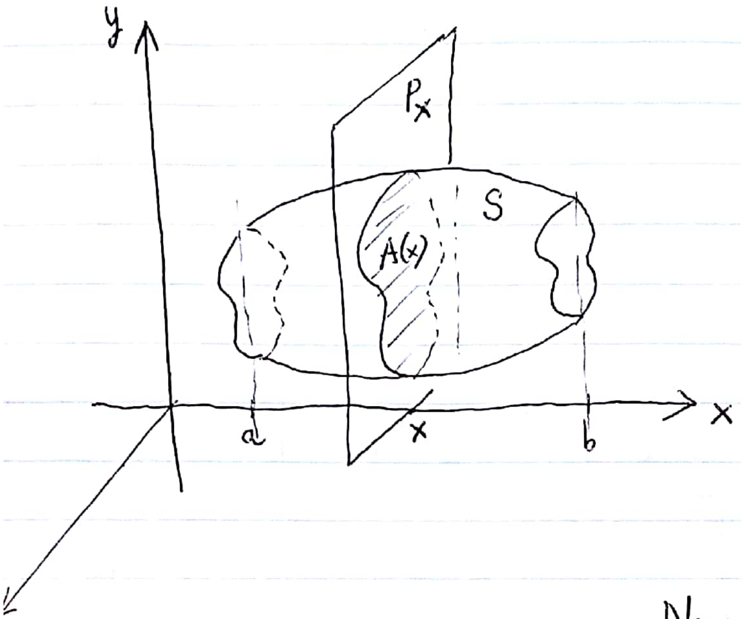
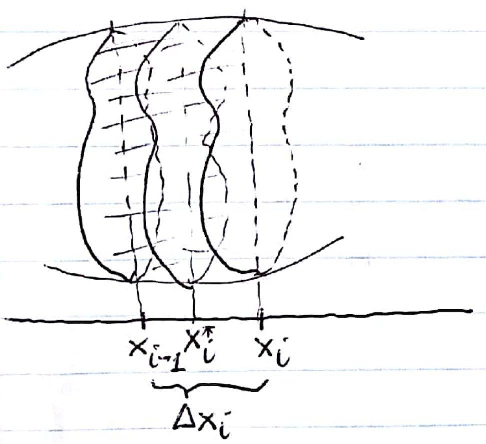
VOLUME



To compute the volume of a solid S we proceed as follows.

Let $A(x)$ be the area of the cross-section of S in the plane P_x perpendicular to the x-axis and passing through the point x, where $a \le x \le b$. The cross-sectional area $A(x)$ will vary as x increases from a to b.



Now for a partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ we can divide S into n "slabs" by using the planes $P_{x_0}, P_{x_1}, \ldots, P_{x_n}$ to slice the solid. Let S_i be the slab that lies between the planes $P_{x_{i-1}}$ and P_{x_i}.

If we choose a sample point $x_i^* \in [x_{i-1}, x_i]$ then we can approximate the volume of S_i as

$$V(S_i) \approx A(x_i^*)\Delta x_i.$$

Therefore we can approximate the volume of S as

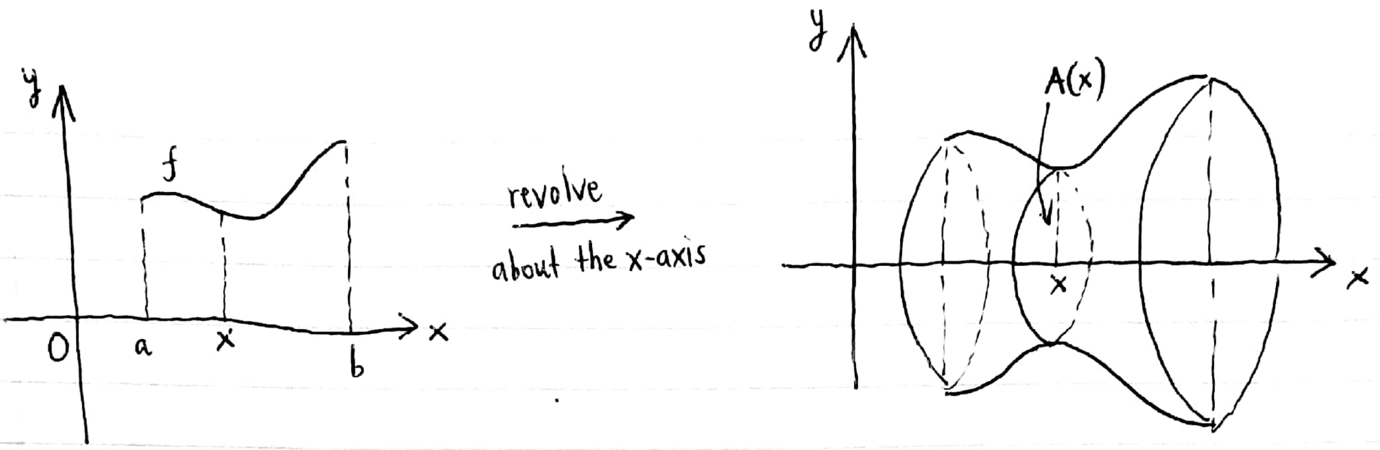
$$V(S) = \sum_{i=1}^{n} V(S_i) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i.$$

Taking the limit as $\|P\| \to 0$ we obtain

$$(1) \qquad \boxed{V(S) = \int_a^b A(x)\,dx}$$

Solids of Revolution: Disk Method

Suppose that $f \geqslant 0$ and is continuous on $[a, b]$. If we revolve about the x-axis the region bounded by the graph of f and the x-axis, we obtain a solid.



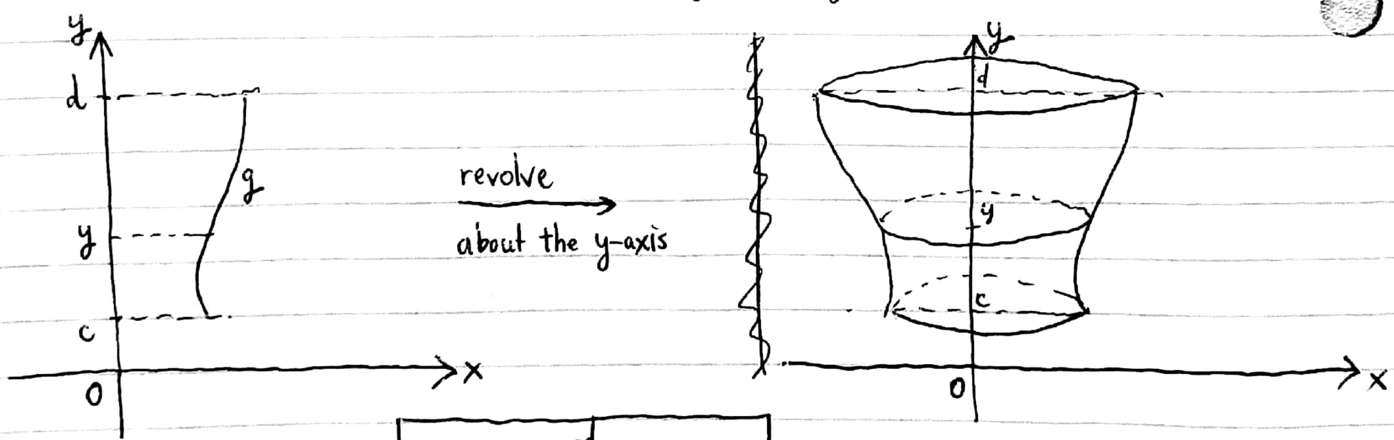
We can obtain the volume of the solid of revolution using formula (1). In this case the cross-section at x is a disk of radius $f(x)$, therefore

$$A(x) = \pi[f(x)]^2.$$

So the volume is

$$(2) \qquad \boxed{V = \pi\int_a^b (f(x))^2\,dx}$$

We can interchange the roles played by x and y. By revolving a continuous function $x = g(y) \geqslant 0$ about the y-axis we obtain a solid with volume given by

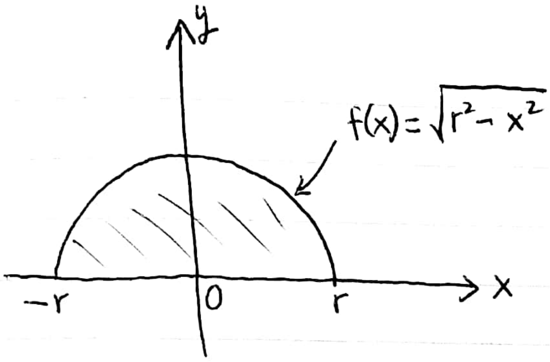


$$(3) \qquad \boxed{V = \pi\int_c^d (g(y))^2\,dy}$$

__Example__. Find the volume of the sphere of radius r.

__Soln__. A sphere of radius r can be obtained by revolving about the x-axis the region below the graph of

$$f(x) = \sqrt{r^2 - x^2}, \qquad -r \leq x \leq r.$$



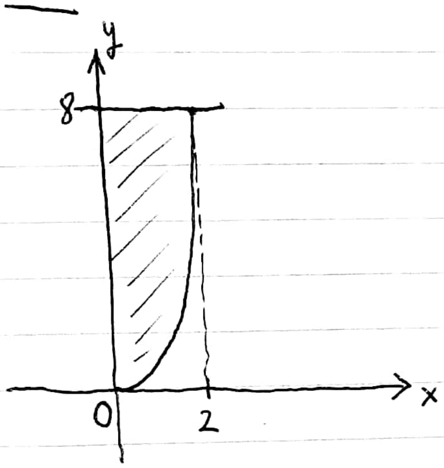
Therefore by (2) the volume is given by

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = \frac{4\pi r^3}{3}. \qquad \square$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Soln.



Since we rotate about the y-axis we need to integrate with respect to y. From

$$y = x^3 \Rightarrow x = \sqrt[3]{y}.$$

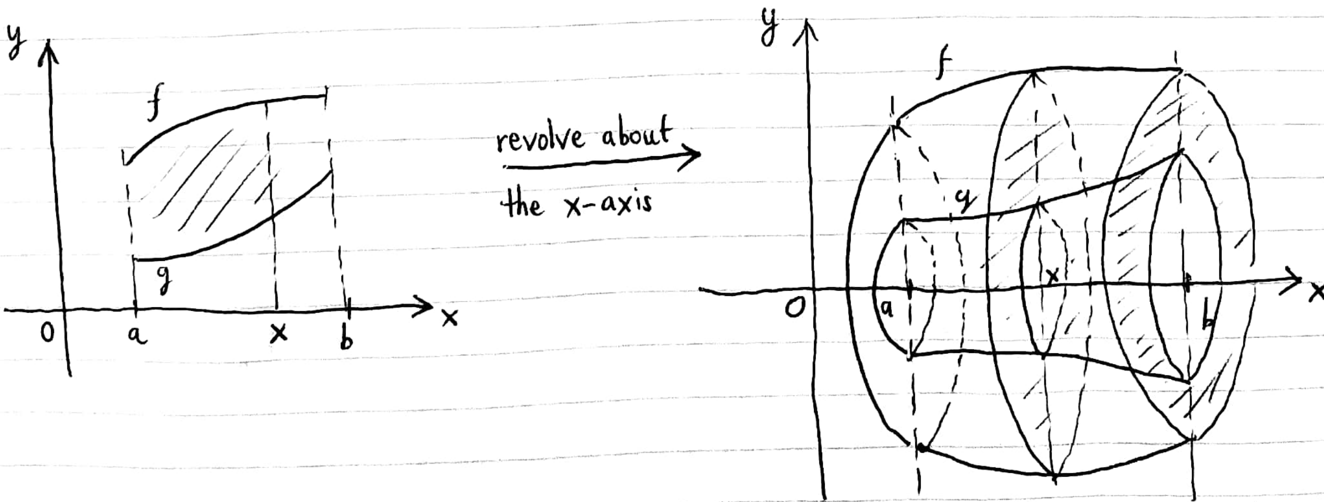
Thus by (3) the volume is given by

$$V = \pi \int_0^8 \left(\sqrt[3]{y}\right)^2 dy = \pi \frac{3}{5} y^{5/3}\Big|_0^8$$

$$= \frac{96\pi}{5}. \qquad \square$$

Solids of Revolution: Washer Method.

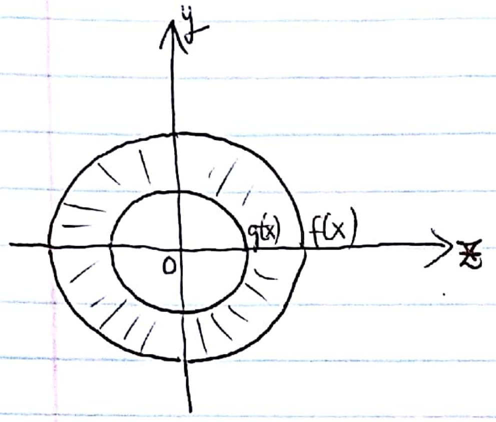
Suppose that f & g are non-negative continuous fns with $g(x) \leq f(x)$ $\forall x \in [a,b]$. If we revolve the region bounded by f & g about the x-axis we obtain a solid



Let us compute the volume of the resulting solid. In this case the cross-section is an annulus, or a "washer" of inner radius $g(x)$ and outer radius $f(x)$, so its area is

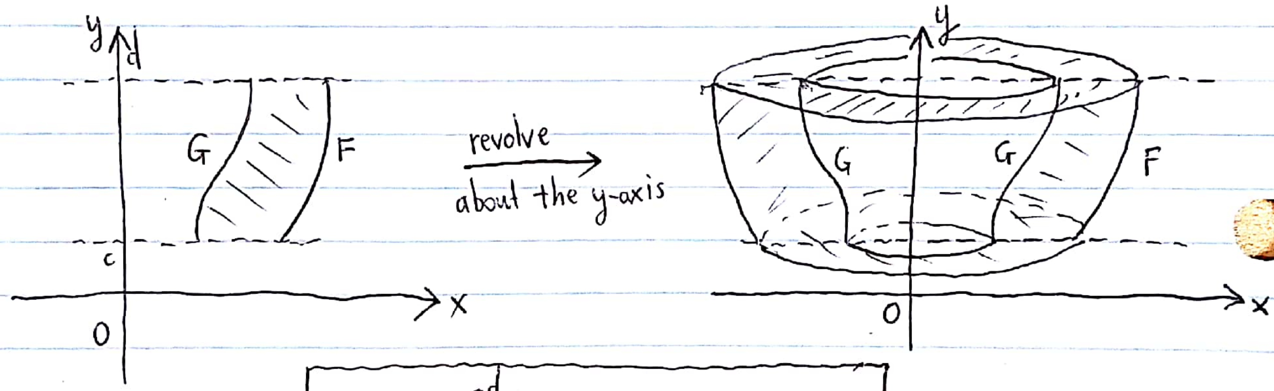
$$A(x) = \pi\left((f(x))^2 - (g(x))^2\right).$$

So by (2) the volume is



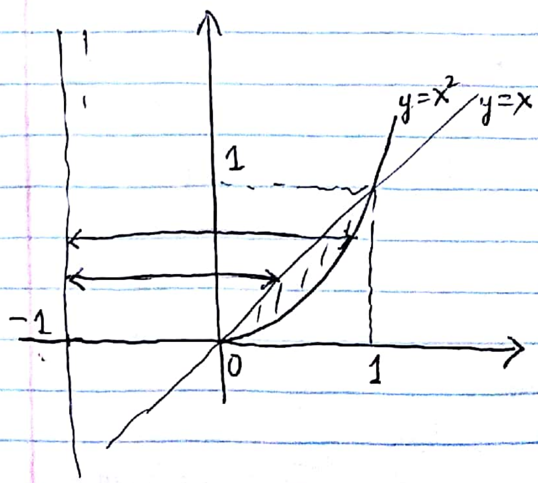
$$\boxed{V = \pi\int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx} \qquad (4)$$

Similarly we can interchange the roles played by x & y.



(5) $$\boxed{V = \pi\int_c^d \left([F(y)]^2 - [G(y)]^2\right)dy}$$

<u>Example</u>. Let R be the region bounded by the curves $y = x$ and $y = x^2$. Find the volume of the solid obtained by rotating R about the line $x = -1$.



<u>Soln</u>. First of all note that

$$x = x^2$$
$$\Leftrightarrow x = 0 \text{ or } x = 1.$$

The region R is illustrated as in the figure.

Notice that in this case we rotate about the line $x=-1$. We first write the curves in terms of y: $x=y$ and $x=\sqrt{y}$.
The cross-section is a washer with inner radius $y+1$ and outer radius $\sqrt{y}+1$.
Therefore by (5) the volume is given by

$$V=\pi\int_0^1\left(\left(\sqrt{y}+1\right)^2-\left(y+1\right)^2\right)dy=\pi\int_0^1\left(2\sqrt{y}-y^2-y\right)dy$$

$$=\pi\left(2\cdot\frac{2}{3}y^{\frac{3}{2}}-\frac{1}{3}y^3-\frac{y^2}{2}\right)\Big|_0^1=\frac{\pi}{2}.$$

$\square$