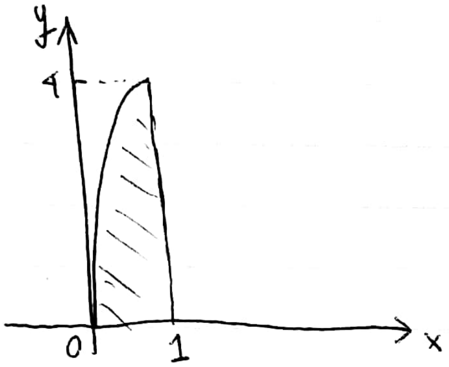
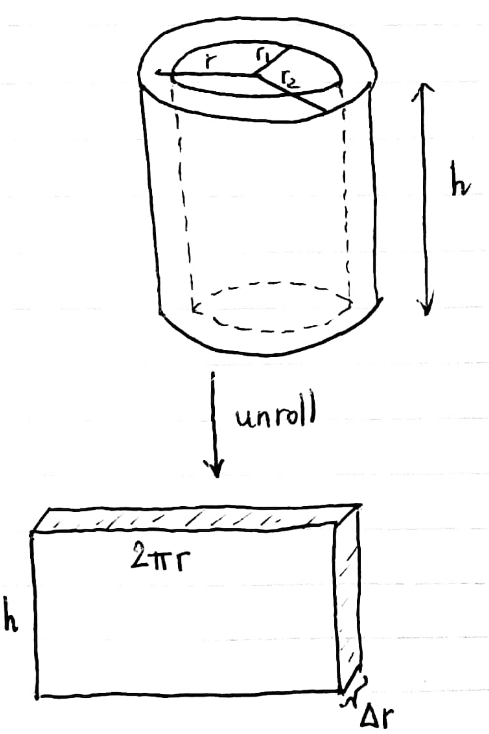
VOLUME 2

SHELL METHOD: Consider the following problem: suppose we want to find the volume of the solid obtained by revolving the region below the curve $y = 5x - x^5$ from $x = 0$ to $x = 1$ about the y-axis.



From the washer method we need to find the inner radius, i.e. we need to express x in terms of y, where $y = 5x - x^5$, which we don't know how to do :(

Now let us introduce a method for these kinds of problems: the shell method. As in the other methods we start with the volume of a simple object: consider a cylindrical shell with inner radius r_1, outer radius r_2, and height h.



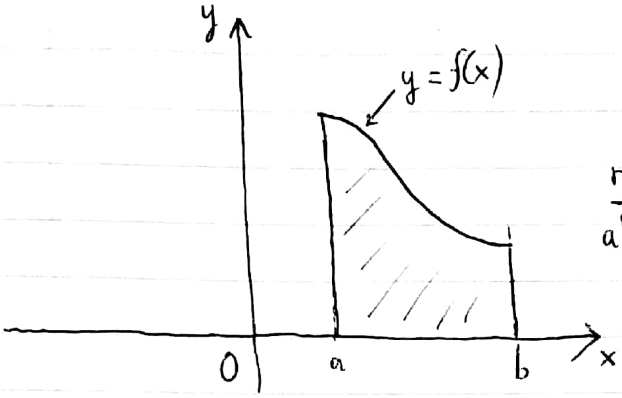
The volume of a cylindrical shell is obtained by subtracting the volume of the inner cylinder from the volume of the outer cylinder

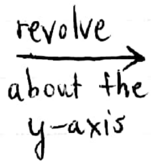
$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi h (r_2^2 - r_1^2) \\ &= \pi h (r_1 + r_2)(r_2 - r_1) \\ &= 2\pi h \underbrace{\left(\frac{r_1 + r_2}{2}\right)}_{r} \underbrace{(r_2 - r_1)}_{\Delta r} \end{aligned}$$

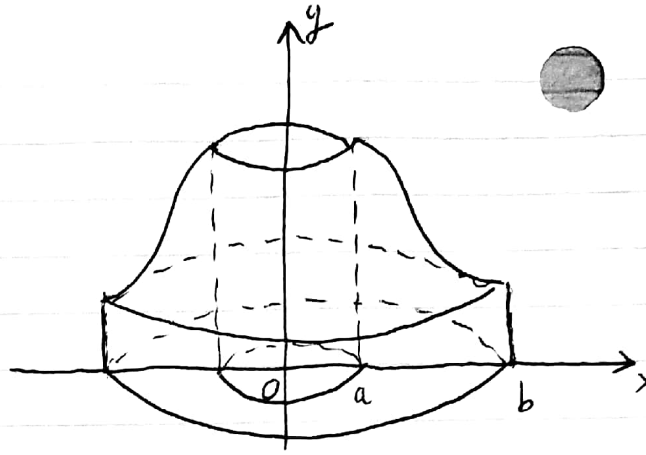
So we obtain $\boxed{V = 2\pi h r \Delta r}$

Another way to obtain the above formula is to "unroll" the cylinder.

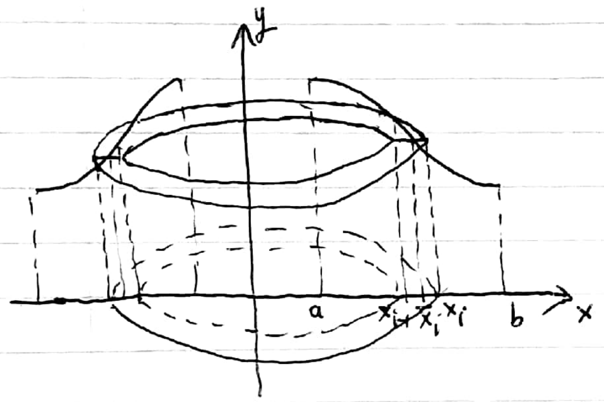
Now let S be the solid obtained by revolving about the y-axis the region below the graph of $y = f(x)$, where $f \geq 0$, continuous, $a \leq x \leq b$.







For a partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$



Let S_i be the cylindrical shell obtained by revolving the rectangle with width Δx_i and height $f(\bar{x}_i) = f\left(\frac{x_{i-1}+x_i}{2}\right)$ about the y-axis.
By the above discussion its volume is given by

$$V_i = 2\pi \bar{x}_i f(\bar{x}_i)\Delta x_i.$$

Then the volume of the solid S can be approximated by the sum of the volumes of these cylindrical shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x_i.$$

Taking limit as $\|P\| \to 0$ we obtain

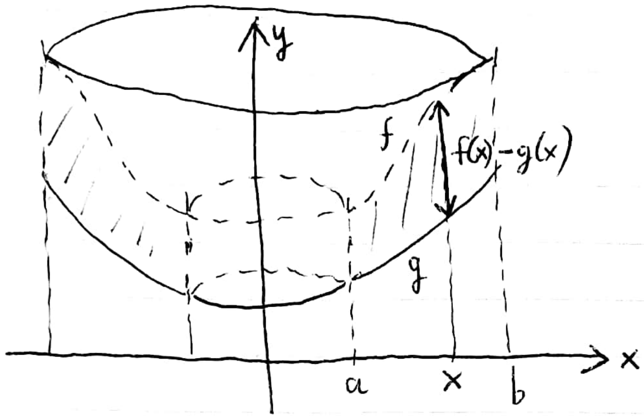
$$\boxed{V = \int_a^b 2\pi x f(x)\,dx}$$

Let us find the volume of the solid we started with:

$$V = \int_0^1 2\pi x(5x - x^5)\,dx = 2\pi \int_0^1 (5x^2 - x^6)\,dx$$

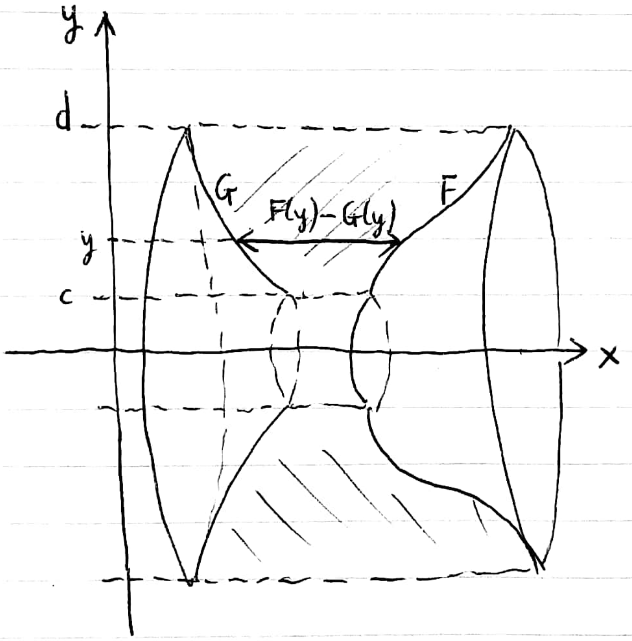
$$= 2\pi\left(\frac{5x^3}{3} - \frac{x^7}{7}\right)\Bigg|_0^1 = \frac{64\pi}{21}.$$

We can generalize the shell method to regions bounded by two functions.



$$\boxed{V = 2\pi \int_a^b x\big(f(x)-g(x)\big)\,dx}$$

revolving the region bounded by
$f(x)$ and $g(x)$, $a \le x \le b$ about the y-axis.



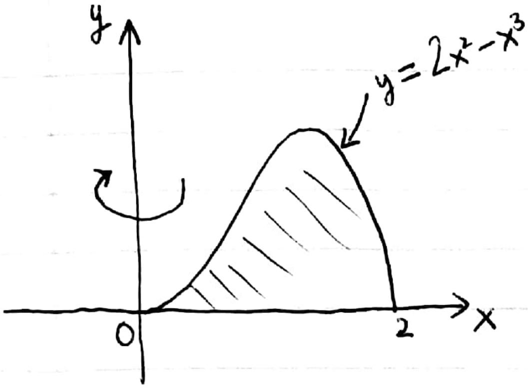
$$\boxed{V = 2\pi \int_c^d y\big(F(y)-G(y)\big)\,dy}$$

revolving the region bounded by
$G(y)$ & $F(y)$, $c \le y \le d$ about
the x-axis

Example. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Soln. First of all observe that

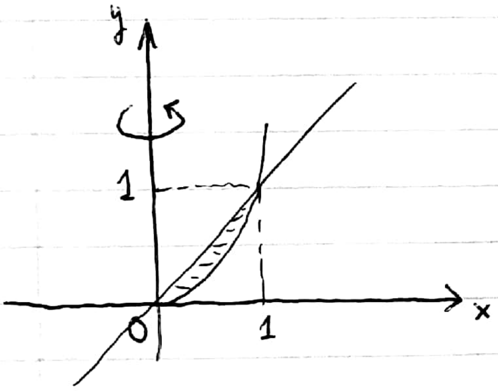
$$2x^2 - x^3 = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$



to apply the washer method we would need to find the inner and outer radii, i.e. to express x in terms of y, which we don't know how to do, so we apply the shell method.

$$V = 2\pi\int_0^2 x(2x^2-x^3)\,dx = 2\pi\left(\frac{x^4}{2}-\frac{x^5}{5}\right)\Big|_0^2 = \frac{16\pi}{5}. \quad \square$$

Example. Find the volume of the solid obtained by rotating about the y-axis the region between $y=x$ & $y=x^2$.



$x=x^2 \Leftrightarrow x=0$ or $x=1$.
We can compute the volume using both the washer method and the shell method

- washer method:

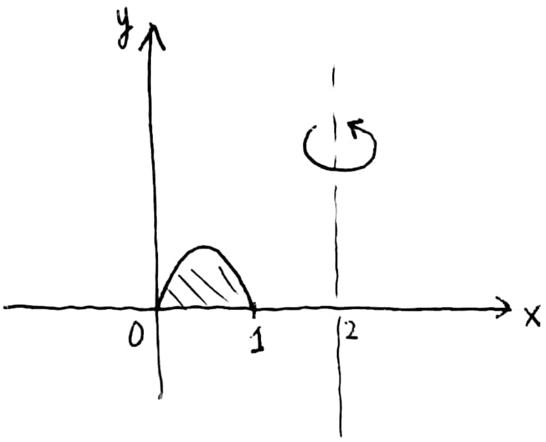
$$V=\pi\int_0^1\left((\sqrt{y})^2-y^2\right)dy = \pi\int_0^1 (y-y^2)\,dy$$
$$=\pi\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_0^1 = \frac{\pi}{6}.$$

- shell method:

$$V=2\pi\int_0^1 x(x-x^2)\,dx = 2\pi\left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{6}. \quad \square$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y=x-x^2$ and $y=0$ about the line $x=2$.

Soln. Note that $x-x^2=0 \Leftrightarrow x=0$ or $x=1$.



to apply the washer method we need to express x in terms of y, where $y = x - x^2$. This is possible but it will complicate the problem since we have to introduce square root. So we use the shell method here.

Observe that for the shell method the radius of the cylinder is $2-x$. Therefore

$$V = 2\pi \int_0^1 (2-x)(x-x^2)\,dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$

$$= 2\pi \left(\frac{x^4}{4} - x^3 + x^2 \right) \Big|_0^1 = \frac{\pi}{2}. \qquad \square$$