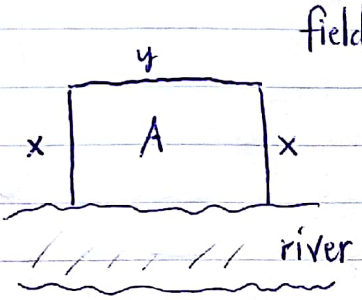
APPLIED OPTIMIZATION PROBLEMS

Problem 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

Soln.



Let x and y be the dimensions of the fence as in the figure (in feet). Let A be the area of the field enclosed by the fence. Then we have

$$A(x,y) = xy.$$

From assumption the total length of the fence is 2400 ft thus

$$2x + y = 2400 \Rightarrow y = 2400 - 2x.$$

Now since x & y are dimensions of the fence, they are non-negative. In particular $x \geq 0$ and $2400 - 2x \geq 0$. Therefore $0 \leq x \leq 1200$. Then we can write A as

$$A = xy = x(2400 - 2x) = 2400x - 2x^2.$$

So our optimization problem is:

$$\boxed{\text{maximize } A(x) = 2400x - 2x^2, \quad 0 \leq x \leq 1200.}$$

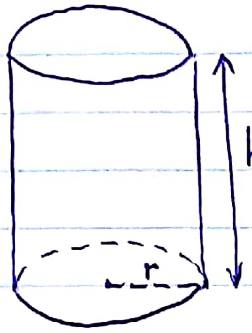
The function $A(x)$ is a polynomial, so it is differentiable everywhere. By the Extreme Value Theorem we know that $A(x)$ has a maximum and a minimum on $[0, 1200]$, which must occur either at the critical points or at the endpoints. To find the critical points we look at

$$A'(x) = 2400 - 4x$$

Then $A'(x) = 0 \Leftrightarrow x = 600 \in [0, 1200]$. Since $A(0) = 0$, $A(1200) = 0$, $A(600) = 720{,}000$, the maximum value of $A(x)$ is $720{,}000$, occurs when $x = 600$(ft) and $y = 2400 - 2x = 1200$(ft). $\square$

Problem 2 A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Soln. Let r be the radius and h be the height of the cylinder (in cm).



The cost of the metal is proportional to the surface area A of the cylinder

$$A = 2\pi r^2 + 2\pi r h$$

To eliminate one of the variables we use the assumption that the volume of the cylinder is $1\,L = 1000\ cm^3$:

$$\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

Plugging $h = 1000/(\pi r^2)$ back in A we get

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

Here our only condition on r is that it is positive. So our optimization problem is

$$\boxed{\text{minimize } A(r) = 2\pi r^2 + \frac{2000}{r}, \quad r > 0}$$

Note that here we cannot invoke the Extreme Value Theorem so we have to investigate $A'(r)$ more carefully. We have

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}.$$

The only critical point in $(0, +\infty)$ is $\sqrt[3]{\frac{500}{\pi}}$. Let's investigate the sign of $A'(r)$.

	0		$\sqrt[3]{\frac{500}{\pi}}$		$+\infty$
$\pi r^3 - 500$		$-$	0	$+$	
r^2	0	$+$	$\vert$	$+$	
$A'(r)$	$-\infty$	$-$	0	$+$	
$A(r)$	$+\infty$	$\searrow$	$A\left(\sqrt[3]{\frac{500}{\pi}}\right)$	$\nearrow$	$+\infty$

So we see that the minimum value of A is

$$A\left(\sqrt[3]{\frac{500}{\pi}}\right) = 2\pi\left(\frac{500}{\pi}\right)^{2/3} + 2000\sqrt[3]{\frac{\pi}{500}}$$

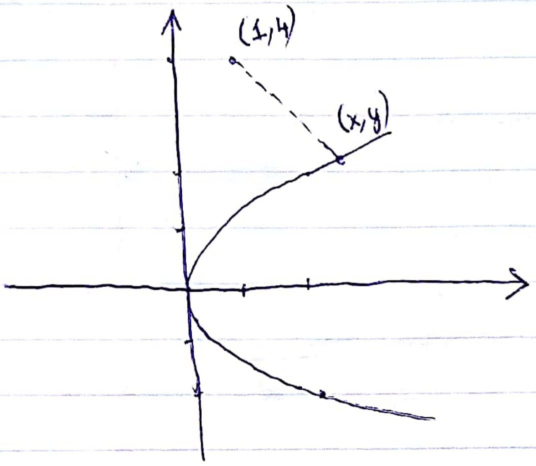
when $r = \sqrt[3]{\frac{500}{\pi}}$ and

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

□

Problem 3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.

Soln.



The distance between the point $(1,4)$ and a point (x,y) is given by

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

If (x,y) is a point on the parabola, then $y^2 = 2x$, or $x = \frac{y^2}{2}$. Thus the distance is

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}.$$

Since $d \geqslant 0$, we can optimize d^2 instead. So our optimization problem is

$$\boxed{\text{minimize } f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2, \quad y \in \mathbb{R}}$$

We have

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right)y + 2(y-4) = y^3 - 8 = (y-2)(y^2+2y+4)$$

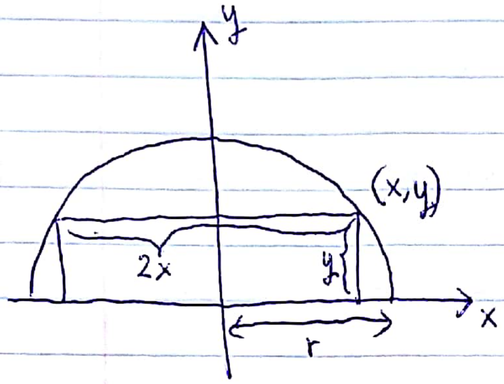
So $f'(y) = 0 \Leftrightarrow y = 2$. So we only have one critical point $y = 2$.

y	$-\infty$		2		$+\infty$
$y-2$		$-$	0	$+$	
$f'(y)$		$-$	0	$+$	
$f(y)$		$\searrow$	$f(2)$	$\nearrow$	

Therefore the minimum value of f is $f(2) = 5$ when $y = 2$ and $x = \frac{y^2}{2} = 2$.
The distance is $d = \sqrt{f(2)} = \sqrt{5}$. $\square$

Problem 4. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Soln.



The word "inscribed" means that the rectangle has two vertices on the semicircle and two vertices on the x-axis.

Let (x,y) be the vertex that lies ~~on~~ in the first quadrant. Then the area of the rectangle is

$$A = 2xy.$$

Since the point (x,y) lies on the semicircle in the first quadrant we have

$$x^2 + y^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2} \quad (\text{because } y \geq 0)$$

So our optimization problem is

$$\boxed{\text{maximize} \quad A = 2x\sqrt{r^2 - x^2}, \quad 0 \leq x \leq r}$$

Since A is continuous on $[0,r]$ by the Extreme Value Theorem we know that A attains its maximum value & its minimum value on $[0,r]$.

Now

$$A'(x) = 2\sqrt{r^2 - x^2} + 2x\left(\frac{-x}{\sqrt{r^2 - x^2}}\right) = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}.$$

Then $A'(x) = 0 \Rightarrow x = \pm \frac{r\sqrt{2}}{2}$. So we have one critical point in $(0,r)$, namely $x = \frac{r\sqrt{2}}{2}$. Then since $A(0) = A(r) = 0$ and $A\left(\frac{r\sqrt{2}}{2}\right) = r^2$, we conclude that the area of the largest inscribed rectangle is r^2. $\square$