LIMITS AT INFINITY.

In this section we investigate the meaning of the expression

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

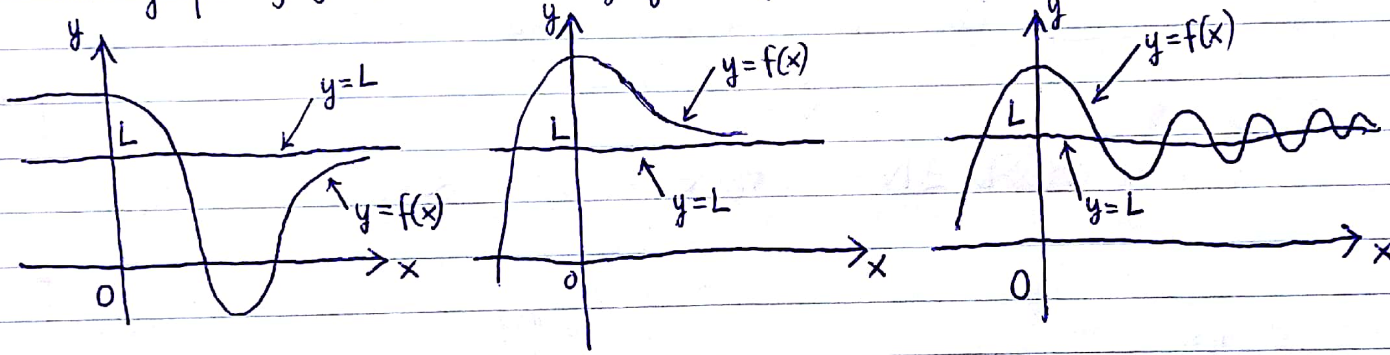
where L is a real number. We read the expression as follows

" the limit of $f(x)$, as x approaches infinity, is L"

or " the limit of $f(x)$, as x becomes infinite, is L"

or " the limit of $f(x)$, as x increases without bound, is L".

The intuitive definition of $\lim_{x\to\infty} f(x) = L$ is we can make the values of $f(x)$ as close to L as we want by choosing x large enough. Geometrically, the graph of f can look roughly as follows.

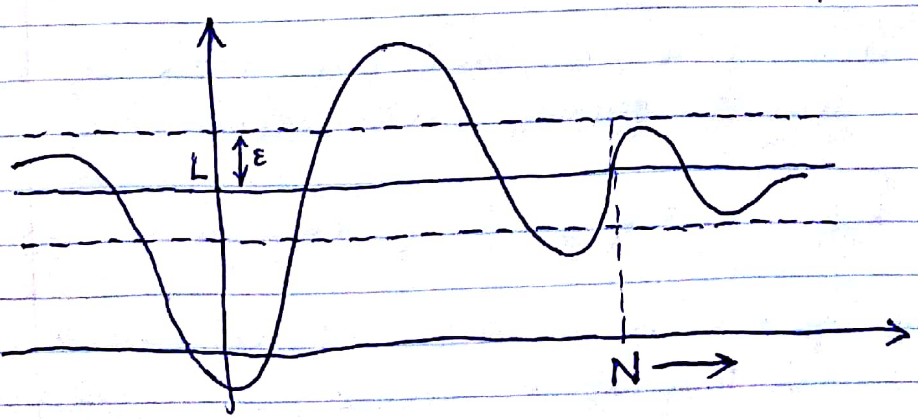


The formal definition of $\lim_{x\to\infty} f(x) = L$ is

Definition. Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x) = L$$

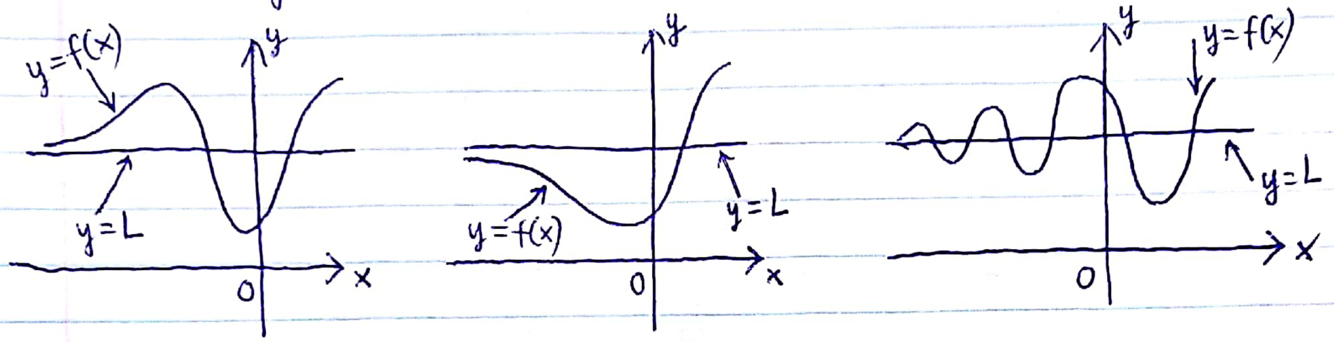
means that $\forall \varepsilon > 0, \exists N \underset{>0}{\cancel{\in \mathbb{R}}}$ s.t. $\forall x \in (a,\infty)$, $x > N \Rightarrow |f(x) - L| < \varepsilon$.



By a similar fashion the expression

$$\boxed{\lim_{x \to -\infty} f(x) = L}$$

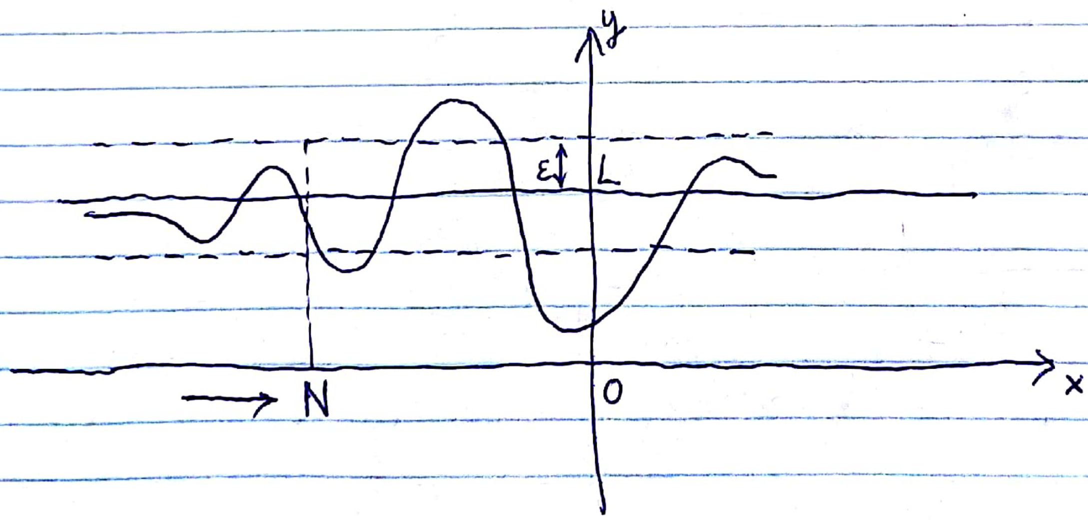
means that we can make $f(x)$ as close to L as we want by choosing x small enough. Geometrically the graph of f looks roughly as follows.



Definition - Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means $\forall \varepsilon > 0 \ \exists N^{<0}$ s.t. $\forall x \in (-\infty, a)$, $x < N \Rightarrow |f(x) - L| < \varepsilon$.



The limit laws remain valid when we replace "$x \to a$" by "$x \to +\infty$" or "$x \to -\infty$", with the following EXCEPTIONS:

$$\lim_{x \to a} x^n = a^n, \ n \in \mathbb{Z}_{>0} \quad ; \quad \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}, \ n \in \mathbb{Z}_{>0}$$

where we cannot replace a by $+\infty$ or $-\infty$. However we will be able to make sense of these ~~un~~ using <u>infinite limits</u>.

Example. Show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Proof. Let $\varepsilon > 0$, we need to find N so that $x > N \Rightarrow \frac{1}{|x|} < \varepsilon$.

Simply choose $N = \frac{1}{\varepsilon}$, then if $x > N > 0 \Rightarrow \frac{1}{x} < \frac{1}{N} = \varepsilon$, (since $x > 0$, $|x| = x$), as required. $\square$

Exercise. Show that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Using the limit laws we obtain the following

Thm. If $r > 0$ is a rational number, then
$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$
If $r > 0$ is a rational number such that x^r is defined for all x, then
$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Example. Evaluate the limit
$$\lim_{x\to\infty} \frac{5x^{5/3} - 7x^{1/3} + 8}{9x^{5/3} + 5x^{2/3} + 2x}$$
$$= \lim_{x\to\infty} \frac{x^{5/3}\left(5 - \frac{7}{x^{4/3}} + \frac{8}{x^{5/3}}\right)}{x^{5/3}\left(9 + \frac{5}{x} + \frac{2}{x^{4/3}}\right)}$$
$$= \frac{\lim_{x\to\infty}\left(5 - \frac{7}{x^{2/3}} + \frac{8}{x^{5/3}}\right)}{\lim_{x\to\infty}\left(9 + \frac{5}{x} + \frac{2}{x^{2/3}}\right)} = \frac{\lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{7}{x^{2/3}} + \lim_{x\to\infty} \frac{8}{x^{5/3}}}{\lim_{x\to\infty} 9 + \lim_{x\to\infty} \frac{5}{x} + \lim_{x\to\infty} \frac{2}{x^{2/3}}}$$
$= \frac{5}{9}$. (where we use the above theorem to conclude that
$$\lim_{x\to\infty} \frac{7}{x^{2/3}} = \lim_{x\to\infty} \frac{8}{x^{5/3}} = \lim_{x\to\infty} \frac{5}{x} = \lim_{x\to\infty} \frac{2}{x^{2/3}} = 0.)$$ $\square$

Definition. The line $y = L$ is called a <u>horizontal asymptote</u> of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) = L$$

Example. Find the horizontal & vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

Soln. Observe that

$$\lim_{x\to(5/3)^+} \frac{\sqrt{2x^2+1}}{3x-5} = +\infty$$ because $\sqrt{2x^2+1}$ is always positive $3x-5 \to 0^+$ when $x \to \left(\frac{5}{3}\right)^+$.

$$\lim_{x\to(5/3)^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$$ because $\sqrt{2x^2+1}$ is always positive and $3x-5 \to 0^-$ when $x \to (5/3)^-$.

So $x = \frac{5}{3}$ is a ~~horizontal~~ vertical asymptote.

To find the horizontal asymptote we compute

- $$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3}$$
because $x > 0$, $|x| = x$

- $$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$
$$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}.$$
because $x < 0$, $|x| = -x$

Thus we have two horizontal asymptotes $y = \frac{\sqrt{2}}{3}$ and $y = \frac{-\sqrt{2}}{3}$. $\square$