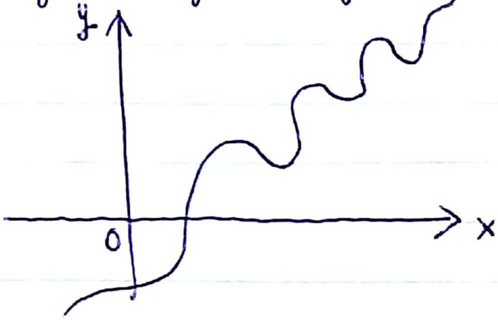
Infinite Limits at Infinity.

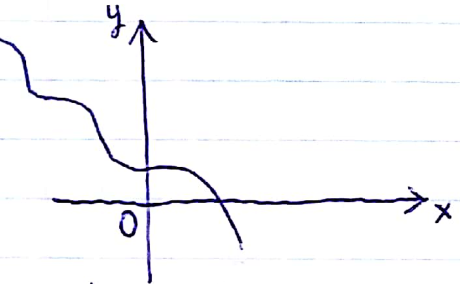
The notation

$$\lim_{x\to\infty} f(x) = \infty$$

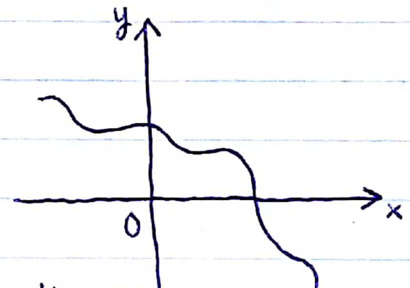
is used to indicate that the values of $f(x)$ become large as x becomes large. In other words, we can make the values of $f(x)$ as large as we want by choosing x large enough. Pictorially, the graph of f looks like



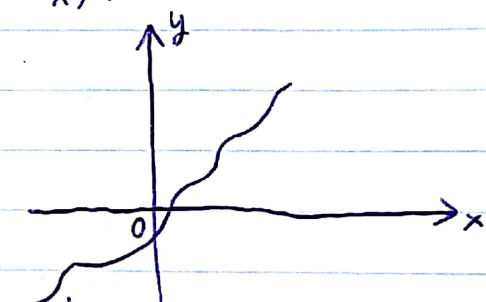
Similarly we also have $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$.



$\lim_{x\to-\infty} f(x) = \infty$



$\lim_{x\to+\infty} f(x) = -\infty$

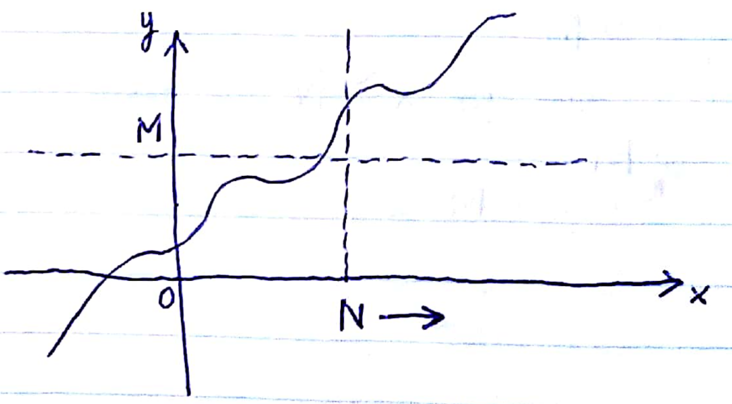


$\lim_{x\to-\infty} f(x) = -\infty$

Definition. Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that $\forall\, M>0\ \exists\, N>0$ s.t. $\forall\, x\in(a,\infty)$, $x>N \Rightarrow f(x)>M$



Exercise. give formal definitions of $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$.

Example. Show that $\lim_{x\to\infty} x^3 = \infty$.

Proof. Given $M>0$ we need to choose $N>0$ s.t. if $x>N$, then $x^3>M$.
Simply choose $N=\sqrt[3]{M}$. Then $x>N \Rightarrow x^3>N^3=M$, as required. $\square$

Example. Find $\lim_{x\to\infty}(x^4-x^3)$.

Soln. Note that we cannot write

$$\lim_{x\to\infty}(x^4-x^3) = \lim_{x\to\infty} x^4 - \lim_{x\to\infty} x^3 = \infty-\infty \quad ✗$$

because both limits do not exist so the limit law does not apply. However we can write

$$\lim_{x\to\infty}(x^4-x^3) = \lim_{x\to\infty} x^3(x-1)$$

Now since $\lim_{x\to\infty} x^3=\infty$ and $\lim_{x\to\infty}(x-1)=\infty$ it is natural to expect that

$\lim_{x\to\infty} x^3(x-1)=\infty$. Let's provide a rigorous proof of this fact

> **Lemma.** Suppose that $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to\infty} g(x)=\infty$, then
> $$\lim_{x\to\infty} f(x)g(x)=\infty$$

Proof. Let $M>10$ be given (in fact you can replace 10 by any number bigger than 1). Then by assumption $\exists\, N_1>0$ and $N_2>0$ s.t.

$$x>N_1 \Rightarrow f(x)>M$$
$$x>N_2 \Rightarrow g(x)>M.$$

Let $N=\max\{N_1,N_2\}>0$, then for $x>N$ we have

$$f(x)g(x) > M^2 > M,$$

because $M>1$

as required. $\square$

Example Find the limit $\lim_{x\to\infty} \frac{x^3+1}{-x^2+10}$.

Soln We have
$$\lim_{x\to\infty} \frac{x^3+1}{-x^2+10} = \lim_{x\to\infty} \frac{x^3(1+\frac{1}{x^3})}{x^2(-1+\frac{10}{x^2})} = \lim_{x\to\infty} \frac{x(1+\frac{1}{x^3})}{-1+\frac{10}{x^2}}$$

Since $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} \frac{1+\frac{1}{x^3}}{-1+\frac{10}{x^2}} = -1$ the resulting limit is $-\infty$.

If you are not convinced by the argument we can prove it rigorously

Lemma. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to+\infty} g(x) = L < 0$, then
$$\lim_{x\to\infty} f(x)g(x) = -\infty$$

Proof. Recall that $\lim_{x\to\infty} f(x)g(x) = -\infty$ means that

$\forall M<0\ \exists N>0$ s.t. $x>N \Rightarrow f(x)g(x) < M$.
So let $M<0$ be given. Since $L<0$ there exists $\varepsilon>0$ s.t. $L+\varepsilon<0$.
Since $\lim_{x\to\infty} g(x) = L$ there exists $N_1>0$ s.t. $x > N_1 \Rightarrow g(x) < L+\varepsilon < 0$.

Because $M<0$ and $L+\varepsilon<0$, we have $\frac{M}{L+\varepsilon} > 0$. Since $\lim_{x\to\infty} f(x) = \infty$

there exists $N_2>0$ s.t. $x>N_2 \Rightarrow f(x) > \frac{M}{L+\varepsilon}$. Now let $N = \max\{N_1, N_2\} > 0$. Then
$$x>N \Rightarrow f(x)g(x) \underset{\text{because } g<0}{<} (L+\varepsilon)\frac{M}{L+\varepsilon} = M,$$

as required. $\square$

Exercise. Show that if $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to+\infty} (-f(x)) = -\infty$.

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Example. Compute $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

Soln. We have

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

$$=0$$

because $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}+x\right)=\infty$. $\square$

Again you can try to prove the following result rigorously.

> **Lemma.** If $\lim\limits_{x\to\infty} f(x)=\infty$, then $\lim\limits_{x\to\infty}\dfrac{1}{f(x)}=0$.

Proof. Let $\varepsilon>0$ be given, then $\exists\, M>0$ so that $\dfrac{1}{M}<\varepsilon$.

Since $\lim\limits_{x\to\infty} f(x)=\infty$ $\exists\, N>0$ s.t. $x>N \Rightarrow f(x)>M$.

Then $\forall x>N$, we have $\dfrac{1}{f(x)}<\dfrac{1}{M}<\varepsilon$, as required. $\square$