INDETERMINATE FORMS & L'HOSPITAL'S RULE.

A limit of the form
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$
where both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ is called an <u>indeterminate form of type $\frac{0}{0}$</u>. It is not obvious how to evaluate those types of limits because the limit law does not apply here. (Recall that $\lim \frac{f}{g} = \frac{\lim f}{\lim g}$ only applies if $\lim f$ & $\lim g$ exist and $\lim g \neq 0$.)

<u>E.g.</u> $\lim_{x\to 1} \frac{\ln x}{x-1}$, $\lim_{x\to 1} \frac{\sin x}{x}$ etc.

A limit of the form
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$
where both $f(x)\to\infty$ (or $-\infty$) and $g(x)\to\infty$ (or $-\infty$) is called an <u>indeterminate form of type $\frac{\infty}{\infty}$</u>. Again the limit law does not apply here since both limits D.N.E.

<u>E.g.</u> $\lim_{x\to\infty} \frac{\ln x}{x-1}$, $\lim_{x\to\infty} \frac{e^x}{x^3}$ etc.

To evaluate those types of limits we have the following result known as <u>L'Hôpital's Rule</u>

<u>Thm</u> (L'Hôpital's Rule) Suppose f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a (except possibly at a).

IF $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

THEN
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
if the limit on the right side exists (or is $\pm\infty$)

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Remark. L'Hôpital's rule is also valid for one-sided limits and for limits at infinity, i.e. "$x \to a$" can be replaced by "$x \to a^+$" or "$x \to a^-$" or "$x \to \infty$" or "$x \to -\infty$".

Proof (sketch) Use the Cauchy Mean Value Theorem

$$(f(x) - f(a))\, g'(c) = (g(x) - g(a))\, f'(c), \quad c \in (a, x).$$

Since $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ can set $f(a) = g(a) = 0$. So

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example. Evaluate $\lim_{x \to \infty} \frac{e^x}{x^2}$

Soln. We have $\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to \infty} x^2 = \infty$, so L'Hôpital's rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x}.$$

Now since $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} 2x = \infty$ we can apply L'Hôpital's rule again

$$\lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty.$$

$\square$

Exercise. Show that $\lim_{x \to \infty} \frac{e^x}{x^n} = 0$ where n is a positive integer.

In particular we see that as $x \to \infty$ the function e^x increases faster than any polynomial, as is evident from the graphs.

Example. Evaluate $\lim_{x \to \infty} \frac{\ln x}{x^p}$, where p is a positive real number.

Soln Since $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to \infty} x^p = \infty$ we can apply L'Hôpital's rule

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = \lim_{x \to \infty} \frac{\frac{1}{x}}{p x^{p-1}} = \lim_{x \to \infty} \frac{1}{p x^p} = 0.$$

$\square$

So we see that as $x \to \infty$ the function $\ln x$ increases slower than any polynomial.

Example. Find $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$

Soln. By L'Hôpital's rule

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}\frac{2\sec x\sec x\tan x}{6x} = \lim_{x\to 0}\frac{\sec^2 x\tan x}{3x}$$

$$\cancel{\underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}} = \left(\lim_{x\to 0}\sec^2 x\right)\left(\lim_{x\to 0}\frac{\tan x}{3x}\right).$$

Now $\lim\limits_{x\to 0}\sec^2 x = 1$ and

$$\lim_{x\to 0}\frac{\tan x}{3x} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}\frac{\sec^2 x}{3} = \frac{1}{3}.$$

So the limit is $\frac{1}{3}$. $\square$

Example. Find $\lim\limits_{x\to \pi^-} \dfrac{\sin x}{1-\cos x}$

Soln. Note that if we blindly apply L'Hôpital's rule we get

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-}\frac{\cos x}{\sin x} = -\infty.$$

This is WRONG since $\lim\limits_{x\to\pi^-}(1-\cos x) = 2$, so L'Hôpital's rule can't be applied here. The limit is not an indeterminate form and it is simply 0.

This example reminds us that we always need to check the conditions of L'Hôpital's rule before applying it. The same is true for other theorems in this course. $\square$

Indeterminate Products.

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$ (or $-\infty$), then the limit

$$\lim_{x\to a} f(x)g(x)$$

is called an indeterminate form of type $0\cdot\infty$. Again limit laws do not apply here and it is not clear what the limit should be.

Our strategy would be to write

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

Then $\lim_{x\to a} f = 0$, $\lim_{x\to a} \frac{1}{g} = 0$ or $\lim_{x\to a} g = \infty$, $\lim_{x\to a} \frac{1}{f} = \infty$, so we can apply L'Hôpital's rule.

<u>Example</u>. Evaluate $\lim_{x\to 0^+} x^p \ln x$, where p is a positive number.

<u>Soln</u>. This is an indeterminate form $0\cdot(-\infty)$ so we write

$$\lim_{x\to 0^+} x^p \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^p}} = \lim_{x\to 0^+} \frac{\ln x}{x^{-p}} \underset{\left(\frac{-\infty}{\infty}\right)}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-px^{-p-1}} = \lim_{x\to 0^+} \frac{1}{-px^{-p}}$$

$$= -\frac{1}{p} \lim_{x\to 0^+} x^p = 0. \qquad \square$$

<u>Example</u>. Evaluate $\lim_{x\to -\infty} x^n e^x$, where n is a positive integer.

<u>Soln</u>. Since $\lim_{x\to -\infty} x^n = \pm\infty$ and $\lim_{x\to -\infty} e^x = 0$ this is an indeterminate form $0\cdot\infty$.

Note that if we write

$$x^n e^x = \frac{e^x}{x^{-n}}$$

then it will not help with the limit since the power of x keeps increasing. So we write

$$\lim_{x\to -\infty} x^n e^x = \lim_{x\to -\infty} \frac{x^n}{e^{-x}} \underset{\substack{\uparrow \\ \text{apply L'hôpital n times}}}{=} \lim_{x\to -\infty} \frac{n!}{(-1)^n e^{-x}} = 0. \qquad \square$$

Therefore in general when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Indeterminate Differences.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} (f(x) - g(x))$$

is called an indeterminate form of type $\infty - \infty$. To deal with limits of this type, our strategy is to convert the difference into a quotient and then apply L'Hôpital's rule.

Example. Compute $\lim_{x\to (\frac{\pi}{2})^-} (\sec x - \tan x)$

Soln. Observe that $\lim_{x\to (\frac{\pi}{2})^-} \sec x = \infty$ and $\lim_{x\to (\frac{\pi}{2})^-} \tan x = \infty$ so the limit is an indeterminate form of type $\infty - \infty$. To proceed, we write

$$\lim_{x\to (\frac{\pi}{2})^-} (\sec x - \tan x) = \lim_{x\to (\frac{\pi}{2})^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x\to (\frac{\pi}{2})^-} \frac{1 - \sin x}{\cos x}$$

$$\underset{(\frac{0}{0})}{=} \lim_{x\to (\frac{\pi}{2})^-} \frac{-\cos x}{-\sin x} = 0,$$

as required. $\square$

Indeterminate Powers.

Now we investigate limits of the form

$$\lim_{x\to a} f(x)^{g(x)}$$

We have three indeterminate forms 0^0, ∞^0, 1^∞. The reason these forms are indeterminate is because we can write

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)}.$$

Then all these indeterminate powers lead to the indeterminate product $0 \cdot \infty$.

To evaluate $\lim_{x\to a} f(x)^{g(x)}$, we either use Logarithm

$$y = f(x)^{g(x)} \Rightarrow \ln y = g(x)\ln f(x)$$

or to write it as an exponential directly

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)}.$$

E.g. Calculate $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

Soln. When $x \to 0^+$ we get the indeterminate form 1^∞. Let
$y = (1+\sin 4x)^{\cot x} \Rightarrow \ln y = \cot x \ln(1+\sin 4x)$.

Now

$$\lim_{x\to 0^+} \cot x \ln(1+\sin 4x) = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \underset{(\frac{0}{0})}{=} \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x}{\sec^2 x(1+\sin 4x)} = 4.$$

Thus we have

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \ln(1+\sin 4x) = 4$$

Taking exponential of both sides and use the continuity of the exponential fn we obtain

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \cancel{e^{\lim_{x\to 0^+} \ln y}} = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^4,$$

as required. □

E.g. Find $\lim_{x\to 0^+} x^x$.

Soln. This is the indeterminate form 0^0. We can write

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}$$

↑ by continuity of the exponential.

Now

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \underset{(\frac{\infty}{\infty})}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} -\frac{1}{x^{-1}} = \lim_{x\to 0^+}(-x) = 0.$$

So $\lim_{x\to 0^+} x^x = 1$. □

E.g. Prove that if $\lim_{x\to\infty} f(x)$ & $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

Proof. Note that replace f with $f+C$ for some constant C will not change f'. So assume $\lim_{x\to\infty} f(x) = L \neq 0$. Then

$$L = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to\infty} \frac{e^x(f(x)+f'(x))}{e^x} = \lim_{x\to\infty}(f(x)+f'(x))$$

↑ $(\frac{\infty}{\infty})$ b/c $L \neq 0$

$$= L + \lim_{x\to\infty} f'(x).$$

So $\lim_{x\to\infty} f'(x) = 0$, as required. □