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Introduction to Number Theory

Problem Set 5 (due Feb 12, 2008)

5.1. a) Let $n = \sum_{j=0}^{d} a_j 10^j$ be an integer with digits $a_d, a_{d-1}, \ldots, a_0$. Assume that $d = 2m + 1$ is odd (if not, add a 0 in front of n). Show that

$$101 \mid n \Leftrightarrow 101 \mid \sum_{j=0}^{m} (-1)^j (a_{2j} + 10a_{2j+1}).$$

The right hand side is the alternating second order sum of digits: group in pairs of two digits, and successively add and subtract these two digit numbers. Use this test to show that 12469135689 is divisible by 101.

b) Calculate

$$\sum_{j=1}^{10} 10^{10^j} \pmod 7.$$

5.2. a) Let $a, b \in \mathbb{Z}$, p a prime. Show that $(a + b)^p \equiv a^p + b^p \pmod p$. *Hint:* One line if you use Little Fermat.

b) Let $a, n \in \mathbb{N}$ be two coprime integers, and assume $a^{n-1} \equiv 1 \pmod n$, but $a^k \not\equiv 1 \pmod n$ for all $1 \leq k \leq n - 2$. Show that n is prime. *Hint:* Show that $a, a^2, \ldots, a^{n-1}$ are $n - 1$ pairwise incongruent mod n and coprime to n. Conclude the claim.

5.3. Find all solutions to $f(x) = x^4 + 3x^3 + 27 \equiv 0 \pmod{5^e}$ for all $e \in \mathbb{N}$.