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# Introduction to Number Theory

## Problem Set 3 (due Jan 29, 2008)

**3.1.** a) Show that there are infinitely many primes of the form $3n + 2$. *Hint:* Assume there are only finitely many. Take a suitable product and show that is must contain another prime of the form $3n + 2$.

b) Why does a similar idea fail for primes of the form $3n + 1$?

c) Let $a, b > 1$ be coprime. Show that $\log_a b$ is irrational.

**3.2.** a) Given that $\log 10 = 2.302585\ldots$ and $\log 2 = 0.693\ldots$, calculate (without aids!) $\log(10^{200})$ and $\log(2 \cdot 10^{200})$. Round to the nearest integer.

b) Given that $1000/461 = 2.169\ldots$, how many primes are there approximately in the interval $[10^{200}, 2 \cdot 10^{200}]$?

c) If you pick randomly an *odd* integer in the interval $[10^{200}, 2 \cdot 10^{200}]$, what is (approximately) the probability that the integer is prime?

**3.3.** a) A positive integer $n > 1$ is called *squarefree*, if it is not divisible by $p^2$ for any prime $p$. A positive integer $n > 1$ is called *squarefull* if $p^2$ divides $n$ for any prime $p$ dividing $n$. For example, $21 = 3 \cdot 7$ is squarefree, but $12 = 2^2 \cdot 3$ is not. $72 = 2^3 \cdot 3^2$ is squarefull, but $50 = 2 \cdot 5^2$ is not. Show that every squarefull number $n$ can *uniquely* be written as $n = a^2 b^3$ where $b$ is squarefree. In other words, show that for every squarefull $n$ there exists a pair $(a, b) \in \mathbb{N}_0^2$ with $b$ squarefree such that $n = a^2 b^3$, and such a pair is unique.

**3.4.** Let $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b, \in \mathbb{Z}\}$ $(\subseteq \mathbb{C})$.

a) Show that the product $xy$ of two elements $x, y \in \mathbb{Z}[\sqrt{-5}]$ is again in $\mathbb{Z}[\sqrt{-5}]$.

b) Verify for yourself that $\mathbb{Z}[\sqrt{5}]$ is a domain by checking all axioms. *Hint:* They follow from the respective axioms in $\mathbb{Z}$ or in $\mathbb{C}$. Do not write this part up.

c) If $x = a + bi \in \mathbb{Z}[\sqrt{-5}]$, show that $|x|^2 = a^2 + 5b^2 \in \mathbb{Z}$. Find all elements $x \in \mathbb{Z}[\sqrt{-5}]$ with $|x|^2 \in \{1, 2, 3\}$.

d) Show that $x \in \mathbb{Z}[\sqrt{-5}]$ is a unit if and only if $|x|^2 = 1$, and write down all units.

e) Show that the four elements $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ are all irreducible. *Hint:* Assume you have a factorization, for example $2 = xy$ with $x, y \in \mathbb{Z}[\sqrt{-5}]$. Take the square of the absolute value on both sides and conclude that either $x$ or $y$ must be a unit.

f) Show that 6 has two distinct factorizations into irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.